
Improved Regret for Bandit Convex Optimization with Delayed Feedback

Yuanyu Wan^{1,2,3}, Chang Yao^{1,2,3}, Mingli Song^{2,3}, Lijun Zhang^{4,2}

¹School of Software Technology, Zhejiang University, Ningbo, China

²State Key Laboratory of Blockchain and Data Security, Zhejiang University, Hangzhou, China

³Hangzhou High-Tech Zone (Binjiang) Institute of Blockchain and Data Security, Hangzhou, China

⁴National Key Laboratory for Novel Software Technology, Nanjing University, Nanjing, China

{wanyy, changy, brooksong}@zju.edu.cn, zhanglj@lamda.nju.edu.cn

Abstract

We investigate bandit convex optimization (BCO) with delayed feedback, where only the loss value of the action is revealed under an arbitrary delay. Let $n, T, \bar{d}$ denote the dimensionality, time horizon, and average delay, respectively. Previous studies have achieved an $O(\sqrt{n}T^{3/4} + (n\bar{d})^{1/3}T^{2/3})$ regret bound for this problem, whose delay-independent part matches the regret of the classical non-delayed bandit gradient descent algorithm. However, there is a large gap between its delay-dependent part, i.e., $O((n\bar{d})^{1/3}T^{2/3})$, and an existing $\Omega(\sqrt{dT})$ lower bound. In this paper, we illustrate that this gap can be filled in the worst case, where $\bar{d}$ is very close to the maximum delay d . Specifically, we first develop a novel algorithm, and prove that it enjoys a regret bound of $O(\sqrt{n}T^{3/4} + \sqrt{dT})$ in general. Compared with the previous result, our regret bound is better for $d = O((n\bar{d})^{2/3}T^{1/3})$, and the delay-dependent part is tight in the worst case. The primary idea is to decouple the joint effect of the delays and the bandit feedback on the regret by carefully incorporating the delayed bandit feedback with a blocking update mechanism. Furthermore, we show that the proposed algorithm can improve the regret bound to $O((nT)^{2/3} \log^{1/3} T + d \log T)$ for strongly convex functions. Finally, if the action sets are unconstrained, we demonstrate that it can be simply extended to achieve an $O(n\sqrt{T} \log T + d \log T)$ regret bound for strongly convex and smooth functions.

1 Introduction

Online convex optimization (OCO) with delayed feedback [Joulani et al., 2013, Quanrud and Khashabi, 2015] has become a popular paradigm for modeling streaming applications without immediate reactions to actions, such as online advertisement [McMahan et al., 2013] and online routing [Awerbuch and Kleinberg, 2008]. Formally, it is defined as a repeated game between a player and an adversary. At each round t , the player first selects an action $\mathbf{x}_t$ from a convex set $\mathcal{K} \subseteq \mathbb{R}^n$. Then, the adversary chooses a convex function $f_t(\cdot) : \mathbb{R}^n \mapsto \mathbb{R}$, which causes the player a loss $f_t(\mathbf{x}_t)$ but is revealed at the end of round $t + d_t - 1$, where $d_t \geq 1$ denotes an arbitrary delay. The goal of the player is to minimize the regret $\text{Reg}(T) = \sum_{t=1}^T f_t(\mathbf{x}_t) - \min_{\mathbf{x} \in \mathcal{K}} \sum_{t=1}^T f_t(\mathbf{x})$, i.e., the gap between the cumulative loss of the player and that of an optimal fixed action, where T is the number of total rounds.

Over the past decades, plenty of algorithms and theoretical guarantees have been proposed for this problem [Weinberger and Ordentlich, 2002, Langford et al., 2009, Joulani et al., 2013, Quanrud and Khashabi, 2015, Joulani et al., 2016, Héliou et al., 2020, Flaspohler et al., 2021, Wan et al., 2022a,b, Bistritz et al., 2022]. However, the vast majority of them assume that the full information or gradients

of delayed functions are available for updating the action, which is not necessarily satisfied in reality. For example, in online routing [Awerbuch and Kleinberg, 2008], the player selects a path through a given network for some packet, and its loss is measured by the time length of the path. Although this loss value can be observed after the packet arrives at the destination, the player rarely has access to the congestion pattern of the entire network [Hazan, 2016]. To address this limitation, it is natural to investigate a more challenging setting, namely bandit convex optimization (BCO) with delayed feedback, where only the loss value $f_t(\mathbf{x}_t)$ is revealed at the end of round $t + d_t - 1$.

It is well known that in the non-delayed BCO, bandit gradient descent (BGD), which performs the gradient descent step based on a one-point estimator of the gradient, enjoys a regret bound of $O(\sqrt{nT^{3/4}})$ [Flaxman et al., 2005]. Despite its simplicity, without additional assumptions on functions, there does not exist any practical algorithm that can improve the regret of BGD. Therefore, a few studies have proposed to extend BGD and its regret bound into the delayed setting [Héliou et al., 2020, Bistritz et al., 2022]. Specifically, Héliou et al. [2020] first propose an algorithm called gradient-free online learning with delayed feedback (GOLD), which utilizes the oldest received but not utilized loss value to perform an update similar to BGD at each round. Let $d = \max\{d_1, \dots, d_T\}$ denote the maximum delay. According to the analysis of Héliou et al. [2020], GOLD can achieve a regret bound of $O(\sqrt{nT^{3/4}} + (nd)^{1/3}T^{2/3})$, which matches the $O(\sqrt{nT^{3/4}})$ regret of BGD in the non-delayed setting for $d = O(\sqrt{nT^{1/4}})$. Very recently, Bistritz et al. [2022] develop an improved variant of GOLD by utilizing all received but not utilized loss values one by one at each round, and reduce the regret bound to $O(\sqrt{nT^{3/4}} + (n\bar{d})^{1/3}T^{2/3})$,¹ where $\bar{d} = (1/T) \sum_{t=1}^T d_t$ is the average delay. However, there still exists a large gap between the delay-dependent part in the improved bound and an existing $\Omega(\sqrt{dT})$ lower bound [Bistritz et al., 2022]. It remains unclear whether this gap can be filled, especially by improving the existing upper bound.

In this paper, we provide an affirmative answer to this question in the worst case, where $\bar{d}$ is very close to d . Specifically, we first develop a new algorithm, namely delayed follow-the-bandit-leader (D-FTBL), and show that it enjoys a regret bound of $O(\sqrt{nT^{3/4}} + \sqrt{dT})$ in general. Notice that both the $O((nd)^{1/3}T^{2/3})$ and $O((n\bar{d})^{1/3}T^{2/3})$ terms in previous regret bounds [Héliou et al., 2020, Bistritz et al., 2022] can be attributed to the joint effect of the delays, and the one-point gradient estimator, especially its large variance depending on the exploration radius. To improve the regret, besides the one-point gradient estimator, we further incorporate the delayed bandit feedback with a blocking update mechanism, i.e., dividing total T rounds into several equally-sized blocks and only updating the action at the end of each block. Despite its simplicity, there exist two nice properties about the cumulative estimated gradients at each block.

- First, with an appropriate block size, its variance becomes proportional to only the block size without extra dependence on the exploration radius.
- Second, the block-level delay, i.e., the number of blocks waiting for computing the cumulative estimated gradients at each block, is in reverse proportion to the block size.

Surprisingly, by combining these properties, the previous joint effect of the delays and the one-point gradient estimator can be decoupled, which is critical for deriving our regret bound. Compared with the existing results, in the worst case, our regret bound matches the $O(\sqrt{nT^{3/4}})$ regret of the non-delayed BGD for a larger amount of delays, i.e., $d = O(n\sqrt{T})$, and the delay-dependent part, i.e., $O(\sqrt{dT})$, matches the lower bound. Moreover, it is worth noting that our regret bound actually is better than that of Bistritz et al. [2022] as long as d is not larger than $O((n\bar{d})^{2/3}T^{1/3})$, which even covers the case with $\bar{d} = O(1)$ partially. To the best of our knowledge, this is the first work that shows the benefit of the blocking update mechanism in delayed BCO, though it is commonly utilized to develop projection-free algorithms for efficiently dealing with complicated action sets [Zhang et al., 2019, Garber and Kretzu, 2020, Hazan and Minasyan, 2020, Wan et al., 2020, 2022c, Wang et al., 2023, 2024b].

Furthermore, we consider the special case of delayed BCO with strongly convex functions. In the non-delayed setting, Agarwal et al. [2010] have shown that BGD can improve the regret from $O(\sqrt{nT^{3/4}})$ to $O((nT)^{2/3} \log^{1/3} T)$ by exploiting the strong convexity. If functions are also smooth and the action set is unconstrained, BGD has been extended to achieve an $O(n\sqrt{T} \log T)$ regret

¹Note that Bistritz et al. [2022] actually only argue a regret bound of $O(nT^{3/4} + \sqrt{n\bar{d}}^{1/3}T^{2/3})$. However, as discussed in our Appendix F, it is not hard to derive this refined bound by tuning parameters more carefully.

bound [Agarwal et al., 2010]. Analogous to these improvements, we prove that our D-FTBL can achieve a regret bound of $O((nT)^{2/3} \log^{1/3} T + d \log T)$ for strongly convex functions, and its simple extension enjoys a regret bound of $O(n\sqrt{T \log T} + d \log T)$ for strongly convex and smooth functions over unconstrained action sets. These regret bounds also match those of BGD in the non-delayed setting for a relatively large amount of delay. Moreover, the $O(d \log T)$ part in these two bounds matches an $\Omega(d \log T)$ lower bound adapted from the easier full-information setting with strongly convex and smooth functions [Weinberger and Ordentlich, 2002].

2 Related work

In this section, we briefly review the related work on online convex optimization (OCO) and bandit convex optimization (BCO), as well as delayed feedback.

2.1 Standard OCO and BCO

If $d_t = 1$ for all $t \in [T]$, OCO with delayed feedback reduces to the standard OCO [Zinkevich, 2003]. Online gradient descent (OGD) [Zinkevich, 2003, Hazan et al., 2007] is one of the most popular algorithm for this problem, which simply updates the action $\mathbf{x}_t$ via a gradient descent step based on $\nabla f_t(\mathbf{x}_t)$. By using appropriate step sizes, OGD can achieve $O(\sqrt{T})$ and $O(\log T)$ regret bounds for convex and strongly convex functions, respectively. Follow-the-regularized-leader (FTRL) [Hazan et al., 2007, Shalev-Shwartz, 2011, Hazan, 2016] is an alternative algorithm, which chooses the new action by minimizing the linear approximation of cumulative loss functions under some regularization. With appropriate regularization, FTRL achieves the same $O(\sqrt{T})$ and $O(\log T)$ regret bounds as OGD. Moreover, Abernethy et al. [2008] have presented a lower bound of $\Omega(\sqrt{T})$ for convex functions, and a refined lower bound of $\Omega(\log T)$ for strongly convex functions, which implies that both OGD and FTRL are optimal.

BCO is a special yet more challenging case of OCO, where the player can only receive the loss value $f_t(\mathbf{x}_t)$ at each round t . The first algorithm for BCO is bandit gradient descent (BGD) [Flaxman et al., 2005], which replaces the exact gradient used in OGD with an estimated gradient based on the single loss value (known as the classical one-point gradient estimator). By incorporating the approximation error of gradients into the regret analysis of OGD, Flaxman et al. [2005] establish an $O(\sqrt{n}T^{3/4})$ regret bound for BGD with convex functions. Later, Agarwal et al. [2010] show that BGD enjoys an $O((nT)^{2/3} \log^{1/3} T)$ regret bound for strongly convex functions, and can be extended to achieve an $O(n\sqrt{T \log T})$ regret bound in the special case of unconstrained BCO with strongly convex and smooth functions. Saha and Tewari [2011] develop a new algorithm for BCO with smooth functions, and establish the $O((nT)^{2/3} \log^{1/3} T)$ regret bound without the strongly convex assumption. van der Hoeven et al. [2020] propose novel BCO algorithms, which adaptively improve the previous regret bounds for convex and smooth functions if the norm of the comparator is small. By revisiting the case with strongly convex and smooth functions, several algorithms [Hazan and Levy, 2014, Ito, 2020] have been developed to achieve the $O(n\sqrt{T \log T})$ regret bound in the constrained setting.

Moreover, a series of studies [Bubeck and Eldan, 2016, Hazan and Li, 2016, Bubeck et al., 2017, Lattimore, 2020, Bubeck et al., 2021] have been devoted to designing nearly optimal algorithms, which almost match the $\Omega(n\sqrt{T})$ lower bound for the general BCO [Shamir, 2013] without any additional assumption. However, the running time of their algorithms are either exponential in n and T , or polynomial with a high degree on n and T , which is not suitable for practical large-scale applications. We refer the interested reader to Lattimore [2024] for a comprehensive survey on BCO. Additionally, we notice that BCO is closely related to the zero-order stochastic optimization (ZOSO) problem [Duchi et al., 2015, Bach and Perchet, 2016, Shamir, 2017], where the stochastic values are available for minimizing a fixed loss function. However, ZOSO is less challenging than BCO in the sense that it does not need to deal with time-varying functions and is usually allowed to query the loss value at two points per iteration.

2.2 OCO and BCO with delays

The seminal work of Weinberger and Ordentlich [2002] first considers the case with a fixed delay, i.e., $d_t = d$ for all $t \in [T]$, and proposes a black-box technique that can covert any traditional OCO

algorithm into the delayed setting. The main idea is to maintain d instances of the traditional algorithm, and alternately utilize these instances to generate the new action. If the regret of the traditional algorithm is bounded by $\text{Reg}(T)$, this technique can achieve an $d \text{Reg}(T/d)$ regret bound. Moreover, there exist $\Omega(\sqrt{dT})$ and $\Omega(d \log T)$ lower bounds for convex functions, and strongly convex and smooth functions, respectively [Weinberger and Ordentlich, 2002]. However, the delays are not always fixed in practice, and its space complexity is d times as much as that of the traditional algorithm, which could be prohibitively resource-intensive. Although Joulani et al. [2013] have generalized this technique to deal with arbitrary delays, the space complexity remains high. Besides these black-box techniques, there exists a surge of interest in developing and analyzing specialized algorithms for delayed OCO [Langford et al., 2009, McMahan and Streeter, 2014, Quanrud and Khashabi, 2015, Joulani et al., 2016, Li et al., 2019, Flaspohler et al., 2021, Wan et al., 2022a,b, 2023, 2024], which do not require additional computational resources.

Despite the great flourish of research on OCO with delays and BCO, delayed BCO has rarely been investigated. GOLD [Héliou et al., 2020] is the first algorithm for this problem, which originally has the $O(\sqrt{n}T^{3/4} + (nd)^{1/3}T^{2/3})$ regret, and is further refined to enjoy the $O(\sqrt{n}T^{3/4} + (nd)^{1/3}T^{2/3})$ regret [Bistriz et al., 2022]. However, Bistriz et al. [2022] also present an unmatched lower bound of $\Omega(\sqrt{dT})$. Although two recent advances in a more complicated bandit non-stochastic control problem [Gradu et al., 2020, Sun et al., 2023] provide some intermediate results about OGD and FTRL with the delayed bandit feedback, they focus on the case with a fixed delay and can only recover the $O(\sqrt{n}T^{3/4} + (nd)^{1/3}T^{2/3})$ regret in general. In this paper, we take one further step toward understanding the effect of arbitrary delays on BCO by establishing improved regret bounds such that the delay-independent part is equal to the regret of BGD, and the delay-dependent part matches the lower bound in the worst case. Moreover, we notice that although the block-box technique of Joulani et al. [2013] can also convert BGD into the delayed setting, it only achieves an $O(\sqrt{nd}^{1/4}T^{3/4})$ regret bound for convex functions, which is much worse than that of GOLD and our algorithm.

3 Main results

In this section, we first introduce the necessary preliminaries including definitions, assumptions, and an algorithmic ingredient. Then, we present our improved algorithm for BCO with delayed feedback, as well as the corresponding theoretical guarantees.

3.1 Preliminaries

We first recall two standard definitions about the smoothness and strong convexity of functions [Boyd and Vandenberghe, 2004].

Definition 1. A function $f(\mathbf{x}) : \mathbb{R}^n \rightarrow \mathbb{R}$ is called β -smooth over $\mathcal{K}$ if for all $\mathbf{x}, \mathbf{y} \in \mathcal{K}$, it holds that $f(\mathbf{y}) \leq f(\mathbf{x}) + \langle \nabla f(\mathbf{x}), \mathbf{y} - \mathbf{x} \rangle + \frac{\beta}{2} \|\mathbf{y} - \mathbf{x}\|_2^2$.

Definition 2. A function $f(\mathbf{x}) : \mathbb{R}^n \rightarrow \mathbb{R}$ is called α -strongly convex over $\mathcal{K}$ if for all $\mathbf{x}, \mathbf{y} \in \mathcal{K}$, it holds that $f(\mathbf{y}) \geq f(\mathbf{x}) + \langle \nabla f(\mathbf{x}), \mathbf{y} - \mathbf{x} \rangle + \frac{\alpha}{2} \|\mathbf{y} - \mathbf{x}\|_2^2$.

Note that as proved by Hazan and Kale [2012], any α -strongly convex function $f(\mathbf{x}) : \mathbb{R}^n \mapsto \mathbb{R}$ over the convex set $\mathcal{K}$ ensures that

$$\frac{\alpha}{2} \|\mathbf{x} - \mathbf{x}^*\|_2^2 \leq f(\mathbf{x}) - f(\mathbf{x}^*) \quad (1)$$

for any $\mathbf{x} \in \mathcal{K}$, where $\mathbf{x}^* = \text{argmin}_{\mathbf{x} \in \mathcal{K}} f(\mathbf{x})$.

Then, following previous studies on BCO [Flaxman et al., 2005, Héliou et al., 2020, Garber and Kretzu, 2020, 2021], we introduce some common assumptions.

Assumption 1. The convex set $\mathcal{K}$ is full-dimensional and contains the origin, and there exist two constants $r, R > 0$ such that $r\mathcal{B}^n \subseteq \mathcal{K} \subseteq R\mathcal{B}^n$, where $\mathcal{B}^n$ denotes the unit Euclidean ball centered at the origin in $\mathbb{R}^n$.

Assumption 2. All loss functions are G -Lipschitz over $\mathcal{K}$, i.e., for all $\mathbf{x}, \mathbf{y} \in \mathcal{K}$ and $t \in [T]$, it holds that $|f_t(\mathbf{x}) - f_t(\mathbf{y})| \leq G\|\mathbf{x} - \mathbf{y}\|_2$.

Assumption 3. The absolute value of all loss functions over $\mathcal{K}$ are bounded by M , i.e., for all $\mathbf{x} \in \mathcal{K}$ and $t \in [T]$, it holds that $|f_t(\mathbf{x})| \leq M$. Additionally, all loss functions are chosen beforehand, i.e., the adversary is oblivious.

Finally, we introduce the one-point gradient estimator [Flaxman et al., 2005], which is a standard technique for exploiting the bandit feedback. Given a function $f(\mathbf{x}) : \mathbb{R}^n \mapsto \mathbb{R}$, we can define the δ -smoothed version of $f(\mathbf{x})$ as

$$\hat{f}_\delta(\mathbf{x}) = \mathbb{E}_{\mathbf{u} \sim \mathcal{B}^n} [f(\mathbf{x} + \delta \mathbf{u})] \quad (2)$$

where the parameter $\delta \in (0, 1)$ is the so-called exploration radius. As proved by Flaxman et al. [2005], the δ -smoothed version satisfies the following lemma.

Lemma 1. (Lemma 1 in Flaxman et al. [2005]) Given a function $f(\mathbf{x}) : \mathbb{R}^n \mapsto \mathbb{R}$ and a constant $\delta \in (0, 1)$, its δ -smoothed version $\hat{f}_\delta(\mathbf{x})$ defined in (2) ensures

$$\nabla \hat{f}_\delta(\mathbf{x}) = \mathbb{E}_{\mathbf{u} \sim \mathcal{S}^n} \left[\frac{n}{\delta} f(\mathbf{x} + \delta \mathbf{u}) \mathbf{u} \right]$$

where $\mathcal{S}^n$ denotes the unit Euclidean sphere centered at the origin in $\mathbb{R}^n$.

From Lemma 1, the randomized vector $\frac{n}{\delta} f(\mathbf{x} + \delta \mathbf{u}) \mathbf{u}$, which can be computed by only utilizing a single loss value, is an unbiased estimator of $\nabla \hat{f}_\delta(\mathbf{x})$. Moreover, Flaxman et al. [2005] have also shown that $\hat{f}_\delta(\mathbf{x})$ is close to the original function $f(\mathbf{x})$ over a shrunk set

$$\mathcal{K}_\delta = (1 - \delta/r) \mathcal{K} = \{(1 - \delta/r) \mathbf{x} | \mathbf{x} \in \mathcal{K}\}. \quad (3)$$

Therefore, this one-point estimator can be utilized as a good substitute for the gradient $\nabla f(\mathbf{x})$ in the bandit setting. For example, we notice that at each round t , BGD [Flaxman et al., 2005] first plays an action $\mathbf{x}_t = \mathbf{y}_t + \mathbf{u}_t$, where $\mathbf{y}_t \in \mathcal{K}_\delta$ and $\mathbf{u}_t \sim \mathcal{S}^n$, and then updates $\mathbf{y}_t$ as

$$\mathbf{y}_{t+1} = \Pi_{\mathcal{K}_\delta} \left(\mathbf{y}_t - \frac{\eta_t n}{\delta} f_t(\mathbf{x}_t) \mathbf{u}_t \right) \quad (4)$$

where $\Pi_{\mathcal{K}_\delta}(\mathbf{y}) = \operatorname{argmin}_{\mathbf{x} \in \mathcal{K}_\delta} \|\mathbf{x} - \mathbf{y}\|_2^2$ denotes the projection onto the set $\mathcal{K}_\delta$, and η_t is the step size.

3.2 Our improved algorithm

Before introducing our algorithm, we first briefly discuss the joint effect of the delays and the bandit feedback in GOLD [Héliou et al., 2020], which will provide insights for our improvements. Recall that in the delayed setting, the loss value $f_t(\mathbf{x}_t)$ will be delayed to the end of round $t + d_t - 1$, and thus the player can only receive $\{f_k(\mathbf{x}_k) | k \in \mathcal{F}_t\}$ at the end of round t , where $\mathcal{F}_t = \{k | k + d_k - 1 = t\}$. Since the set $\mathcal{F}_t$ may not contain the round t , the vanilla BGD in (4) is no longer valid. To address this issue, GOLD [Héliou et al., 2020] replaces $f_t(\mathbf{x}_t)$ in (4) with the oldest received but not utilized loss value at the end of round t . Intuitively, the update of this approach is $O(d)$ rounds slower than that of the vanilla BGD, which is analogous to those delayed OCO algorithms. However, due to the use of the one-point gradient estimator, the slower update causes a difference of $O(\eta d n / \delta)$ between its action and that of BGD, and the cumulative difference will bring additional regret of $O(T \eta d n / \delta)$, where a constant step size $\eta_t = \eta$ is discussed for brevity. Note that from the standard analysis of BGD, to control the total exploration cost, the value of $1/\delta$ should be sublinear in T . Therefore, it will amplify the effect of delays, and finally results in the $O(\sqrt{n} T^{3/4} + (nd)^{1/3} T^{2/3})$ regret [Héliou et al., 2020].

To reduce the effect of delays, we propose to incorporate the delayed bandit feedback with a blocking update mechanism [Zhang et al., 2019, Garber and Kretzu, 2020]. Specifically, we divide the total T rounds into T/K blocks, each with K rounds, where T/K is assumed to be an integer without loss of generality. For each block $m \in [T/K]$, we only maintain a preparatory action $\mathbf{y}_m \in \mathcal{K}_\delta$, and play $\mathbf{x}_t = \mathbf{y}_m + \delta \mathbf{u}_t$ with $\mathbf{u}_t \sim \mathcal{S}^n$ at each round t in the block. Due to the randomness of $\mathbf{u}_t$ and the independence of $\mathbf{x}_t$ in the same block, it is not hard to verify that for each block $m \in [T/K]$, the sum of randomized gradients generated by the one-point estimator, i.e., $\nabla_m = \sum_{t=(m-1)K+1}^{mK} \frac{n}{\delta} f_t(\mathbf{x}_t) \mathbf{u}_t$, satisfies (see Lemma 5 presented in Section 3.4 for details)

$$\mathbb{E}[\|\nabla_m\|_2] = O(\sqrt{K n^2 / \delta^2} + K).$$

By using an appropriate block size of $K = O(n^2 / \delta^2)$, this upper bound will be $\mathbb{E}[\|\nabla_m\|_2] = O(K)$. By contrast, without the blocking update mechanism, one can only achieve $\mathbb{E}[\|\nabla_m\|_2] = O(K n / \delta)$. Moreover, we notice that the cumulative estimated gradients ∇_m will be delayed at most $O(d/K)$

Algorithm 1 Delayed Follow-The-Bandit-Leader

```

1: Input:  $\delta, K, \alpha$ , and  $\eta > 0$  if  $\alpha = 0$ 
2: Initialization: set  $\bar{\mathbf{g}}_0 = \mathbf{0}$  and choose  $\mathbf{y}_1 \in \mathcal{K}_\delta$  arbitrarily
3: for  $m = 1, 2, \dots, T/K$  do
4:   for  $t = (m-1)K + 1, \dots, mK$  do
5:     Play  $\mathbf{x}_t = \mathbf{y}_m + \delta \mathbf{u}_t$ , where  $\mathbf{u}_t \sim \mathcal{S}^n$ 
6:     Query  $f_t(\mathbf{x}_t)$ , and receive  $\{f_k(\mathbf{x}_k) | k \in \mathcal{F}_t\}$ 
7:     Update  $\bar{\mathbf{g}}_t = \bar{\mathbf{g}}_{t-1} + \sum_{k \in \mathcal{F}_t} \frac{n}{\delta} f_k(\mathbf{x}_k) \mathbf{u}_k$ 
8:   end for
9:   Set  $\mathcal{R}_m(\mathbf{x}) = \begin{cases} \frac{1}{\eta} \|\mathbf{x} - \mathbf{y}_1\|_2^2 & \text{if } \alpha = 0 \\ \sum_{i=1}^m \frac{K\alpha}{2} \|\mathbf{x} - \mathbf{y}_i\|_2^2 & \text{otherwise} \end{cases}$ 
10:   $\mathbf{y}_{m+1} = \operatorname{argmin}_{\mathbf{x} \in \mathcal{K}_\delta} \{\langle \bar{\mathbf{g}}_{mK}, \mathbf{x} \rangle + \mathcal{R}_m(\mathbf{x})\}$ 
11: end for

```

blocks, because even the last component $\frac{n}{\delta} f_{mK}(\mathbf{x}_{mK}) \mathbf{u}_{mK}$ is available at the end of round $mK + d - 1$.

As a result, one possible approach to determine $\mathbf{y}_m$ for each block is to extend the update rule of GOLD [Héliou et al., 2020] into the block level with $K = O(n^2/\delta^2)$. Combining with previous discussions, it will reduce the effect of delays on the regret from $O(T\eta dn/\delta)$ to

$$O\left(T\eta \frac{d}{K} \left(\sqrt{\frac{Kn^2}{\delta^2}} + K\right)\right) = O(\eta dT)$$

which is good enough for deriving our desired regret bounds. However, it requires a bit complicated procedure to maintain the cumulative estimated gradients for any block that has not been utilized to update the action. For this reason, instead of utilizing this approach, we incorporate FTRL [Hazan et al., 2007, Hazan, 2016] with the delayed bandit feedback and blocking update mechanism, which provides a more elegant way to utilize the delayed information.

Specifically, we initialize $\mathbf{y}_1 \in \mathcal{K}_\delta$ arbitrarily, and use a variable $\bar{\mathbf{g}}_t$ to record the sum of gradients estimated from all received loss values, i.e., $\bar{\mathbf{g}}_t = \sum_{i=1}^t \sum_{k \in \mathcal{F}_i} \frac{n}{\delta} f_k(\mathbf{x}_k) \mathbf{u}_k$. Then, according to FTRL, an ideal action should be selected by minimizing the linear approximation of cumulative loss functions under some regularization, i.e.,

$$\mathbf{y}_{m+1}^* = \operatorname{argmin}_{\mathbf{x} \in \mathcal{K}_\delta} \left\{ \sum_{i=1}^m \langle \nabla_i, \mathbf{x} \rangle + \mathcal{R}_m(\mathbf{x}) \right\} \quad (5)$$

where the regularization is set as $\mathcal{R}_m(\mathbf{x}) = \frac{1}{\eta} \|\mathbf{x} - \mathbf{y}_1\|_2^2$ for convex functions [Hazan, 2016] and $\mathcal{R}_m(\mathbf{x}) = \sum_{i=1}^m \frac{K\alpha}{2} \|\mathbf{x} - \mathbf{y}_i\|_2^2$ for α -strongly convex functions [Hazan et al., 2007]. Unfortunately, due to the effect of delays, the value of $\sum_{i=1}^m \nabla_i$ required by (5) may not be available. To address this limitation, we generate $\mathbf{y}_{m+1}$ by replacing this term with the sum of all available estimated gradients, i.e., $\bar{\mathbf{g}}_{mK}$.²

The detailed procedures are outlined in Algorithm 1, where the input α is the modules of the strong convexity of functions, and it is called delayed follow-the-bandit-leader (D-FTBL).

3.3 Theoretical guarantees

We first present the regret bound of our D-FTBL for convex functions.

Theorem 1. *Under Assumptions 1, 2, and 3, Algorithm 1 with $\alpha = 0$ ensures*

$$\mathbb{E}[\operatorname{Reg}(T)] \leq \underbrace{\frac{4R^2}{\eta} + \frac{\eta T \gamma}{2K}}_{:=A} + \underbrace{\frac{\eta T G}{2} \sqrt{2 \left(\frac{d^2}{K^2} + 4 \right)}}_{:=B} \gamma + \underbrace{3\delta GT + \frac{\delta GRT}{r}}_{:=C} \quad (6)$$

where $\gamma = K \left(\frac{nM}{\delta} \right)^2 + K^2 G^2$.

²From the above discussions, one may replace $\sum_{i=1}^m \nabla_i$ with the sum of all available ∇_i . However, we find that simply utilizing $\bar{\mathbf{g}}_{mK}$ can attain the same regret, though they have a slight difference.

Remark. To help understanding the regret bound in (6), we notice that the term A actually is derived from the expected regret of the ideal action $\mathbf{y}_m^*$ on a sequence of surrogate losses, and the term B is caused by the cumulative distance between our preparatory action $\mathbf{y}_m$ and the ideal one. Additionally, the term C is caused by the exploration error of the one-point gradient estimator. At first glance, it seems that term B suffers a multiplicative joint effect of the maximum delay d and the exploration radius δ due to the existence of γ . However, as discussed before, this joint effect can be decoupled by setting an appropriate block size of $K = O(n^2/\delta^2)$, which allows us to derive an improved regret bound. Specifically, by substituting $\alpha = 0$, $K = n\sqrt{T}$, $\eta = 1/\max\{\sqrt{Td}, \sqrt{n}T^{3/4}\}$, and $\delta = c\sqrt{n}T^{-1/4}$ into (6), where c is a constant such that $\delta < r$, our D-FTBL can enjoy

$$\mathbb{E}[\text{Reg}(T)] \leq O\left(\sqrt{n}T^{3/4} + \sqrt{dT}\right) \quad (7)$$

for convex functions.³ It is tighter than the $O(\sqrt{n}T^{3/4} + (nd)^{1/3}T^{2/3})$ regret of GOLD [Héliou et al., 2020], and matches the $O(\sqrt{n}T^{3/4})$ regret bound of BGD in the non-delayed setting as long as d is not larger than $O(n\sqrt{T})$. Even for $d = \Omega(n\sqrt{T})$, our regret bound is dominated by the $O(\sqrt{dT})$ part, which matches the $\Omega(\sqrt{dT})$ lower bound [Bistritz et al., 2022] in the worst case. Moreover, although the $O(\sqrt{n}T^{3/4} + (nd)^{1/3}T^{2/3})$ regret bound of Bistritz et al. [2022] could benefit from a small average delay, it is also worse than our regret bound when d is not larger than $O((nd)^{2/3}T^{1/3})$.

Then, we establish an improved regret bound for α -strongly convex functions.

Theorem 2. Under Assumptions 1, 2, and 3, if all functions are α -strongly convex, Algorithm 1 with $\alpha > 0$ ensures

$$\mathbb{E}[\text{Reg}(T)] \leq \underbrace{\frac{2\gamma C_T}{\alpha K} + C'_T R \sqrt{\gamma}}_{:=A'} + \underbrace{\frac{GC_T}{\alpha} \sqrt{2\left(\frac{d^2}{K^2} + 4\right)}}_{:=B'} \gamma + 3\delta GT + \frac{\delta GT}{r} \quad (8)$$

where $\gamma = K\left(\frac{nM}{\delta}\right)^2 + K^2 G^2$, $C_T = 1 + \ln T$, and $C'_T = 6 + 4 \ln T$.

Remark. By comparing Theorem 2 with Theorem 1, we find that the strong convexity can be exploited to reduce the expected regret of the ideal action $\mathbf{y}_m^*$ on the surrogate losses, and the cumulative distance between our preparatory action $\mathbf{y}_m$ and the ideal one, i.e., improving terms A and B in (6) to terms A' and B' in (8). By further substituting $\alpha > 0$, $K = (nT)^{2/3} \ln^{-2/3} T$, and $\delta = cn^{2/3}T^{-1/3} \ln^{1/3} T$ into (8), where c is a constant such that $\delta < r$, our D-FTBL can enjoy

$$\mathbb{E}[\text{Reg}(T)] \leq O\left((nT)^{2/3} \log^{1/3} T + d \log T\right) \quad (9)$$

for strongly convex functions. This regret bound is tighter than the above $O(\sqrt{n}T^{3/4} + \sqrt{dT})$ regret bound achieved by only utilizing the convexity condition, and can match the $O((nT)^{2/3} \log^{1/3} T)$ regret bound of BGD in the non-delayed setting as long as d is not larger than $O((nT/\log T)^{2/3})$. Even if $d = \Omega((nT/\log T)^{2/3})$, it is dominated by the $O(d \log T)$ part, which matches the $\Omega(d \log T)$ lower bound [Weinberger and Ordentlich, 2002], and thus cannot be improved. Moreover, different from the case with convex functions, the parameters for achieving the bound in (9) do not require the information of delays.

Furthermore, we consider the unconstrained case, i.e., $\mathcal{K} = \mathbb{R}^n$, with α -strongly convex and β -smooth functions, and extend our D-FTBL to achieve a better regret bound. Specifically, without the boundedness of $\mathcal{K}$, Assumptions 2 and 3 may no longer hold over the entire space [Agarwal et al., 2010]. Therefore, we first introduce a weaker assumption on the Lipschitz continuity, i.e., all loss functions are G -Lipschitz at $\mathbf{0}$. Combining with (1), it is not hard to verify that the fixed optimal action $\mathbf{x}^* = \arg\min_{\mathbf{x} \in \mathbb{R}^n} \sum_{t=1}^T f_t(\mathbf{x})$ satisfies

$$\|\mathbf{x}^*\|_2 \leq \frac{2G}{\alpha}. \quad (10)$$

³One may notice that the step size for achieving this result depends on the maximum delay d , which may be unknown beforehand. Fortunately, as discussed in previous studies [Quanrud and Khashabi, 2015, Wan et al., 2024], there exists a standard solution—utilizing the “doubling trick” [Cesa-Bianchi et al., 1997] to adaptively estimate the maximum delay d and adjust the step size, which can attain the same bound as in (7).

As a result, the player only needs to select actions from the following set

$$\mathcal{K}' = \left\{ \mathbf{x} \in \mathbb{R}^n \mid \|\mathbf{x}\|_2 \leq \frac{2G}{\alpha} \right\} \quad (11)$$

which satisfies Assumption 1 with $r = R = 2G/\alpha$, and it is natural to further assume that all loss functions satisfy Assumptions 2 and 3 over the set $\mathcal{K}'$. Now, we can apply our D-FTBL over the shrink set of $\mathcal{K}'$, i.e.,

$$\mathcal{K}'_\delta = (1 - \delta/r)\mathcal{K}' = \left(1 - \frac{\alpha\delta}{2G}\right)\mathcal{K}' \quad (12)$$

instead of the original $\mathcal{K}_\delta$, and establish the following regret bound.

Theorem 3. *Let $\mathcal{K} = \mathbb{R}^n$. If all loss functions are α -strongly convex and β -smooth over $\mathcal{K}$, and Assumptions 2 and 3 hold over $\mathcal{K}'$ defined in (11), applying Algorithm 1 with $\alpha > 0$ over $\mathcal{K}'_\delta$ defined in (12) ensures*

$$\mathbb{E}[\text{Reg}(T)] \leq \frac{2\gamma C_T}{\alpha K} + \frac{2C'_T G \sqrt{\gamma}}{\alpha} + \frac{GC_T}{\alpha} \sqrt{2 \left(\frac{d^2}{K^2} + 4 \right) \gamma} + \underbrace{\beta \delta^2 T + \frac{\beta \delta^2 GT}{\alpha}}_{:=C'} \quad (13)$$

where $\gamma = K \left(\frac{nM}{\delta} \right)^2 + K^2 G^2$, $C_T = 1 + \ln T$, and $C'_T = 6 + 4 \ln T$.

Remark. By comparing Theorem 3 with Theorem 2, we find that the exploration error of the one-point gradient estimator is reduced, i.e., improving the last two terms in (8) to the term C' in (13). Then, by substituting $\alpha > 0$, $K = n\sqrt{T/\ln T}$, and $\delta = cn^{1/2}T^{-1/4} \ln^{1/4} T$ into (13), where c is a constant such that $\delta < 2G/\alpha$, we can achieve an $O(n\sqrt{T \log T} + d \log T)$ regret bound for strongly convex and smooth functions in the unconstrained case. It is better than the $O((nT)^{2/3} \log^{1/3} T + d \log T)$ regret bound achieved by only utilizing the strong convexity. Moreover, this bound matches the $O(n\sqrt{T \log T})$ regret bound achieved by using BGD in the non-delayed setting as long as d is not larger than $O(n\sqrt{T/\log T})$. Otherwise, it is dominated by the $O(d \log T)$ part, which cannot be improved as discussed before.

3.4 Analysis: proof of Theorem 1

Due to the limitation of space, here we only prove Theorem 1, and the omitted proofs can be found in the appendix. Specifically, let $\tilde{\mathbf{x}}^* = (1 - \delta/r)\mathbf{x}^*$ where $\mathbf{x}^* \in \arg\min_{\mathbf{x} \in \mathcal{K}} \sum_{t=1}^T f_t(\mathbf{x})$, and recall the ideal action defined in (5). As in Lemma 2, we first notice that the expected regret of Algorithm 1 can be bounded by the sum of three parts including the expected regret of ideal actions on some surrogate losses, the cumulative distance between $\mathbf{y}_m$ and the ideal one, and the exploration error of the one-point gradient estimator.

Lemma 2. *Under Assumptions 1 and 2, Algorithm 1 with $\alpha = 0$ ensures*

$$\mathbb{E}[\text{Reg}(T)] \leq \mathbb{E} \left[\sum_{m=1}^{T/K} \langle \nabla_m, \mathbf{y}_m^* - \tilde{\mathbf{x}}^* \rangle + KG \sum_{m=1}^{T/K} \|\mathbf{y}_m - \mathbf{y}_m^*\|_2 \right] + 3\delta GT + \frac{\delta GRT}{r}. \quad (14)$$

Note that the part regarding the exploration error in (14) is exactly the same as the term C in (6). So, we only need to analyze the first two parts in (14). For the first part, we define surrogate losses as $\ell_1(\mathbf{x}) = \langle \nabla_1, \mathbf{x} \rangle + \frac{1}{\eta} \|\mathbf{x} - \mathbf{y}_1\|_2$ and $\ell_m(\mathbf{x}) = \langle \nabla_m, \mathbf{x} \rangle$ for any $m = 2, \dots, T/K$. Combining with (5) for convex functions, it is easy to verify that $\mathbf{y}_{m+1}^* = \arg\min_{\mathbf{x} \in \mathcal{K}_\delta} \sum_{i=1}^m \ell_i(\mathbf{x})$. Then, we introduce the following lemma to bound the regret of $\mathbf{y}_2^*, \dots, \mathbf{y}_{T/K+1}^*$ on $\ell_1(\cdot), \dots, \ell_{T/K}(\cdot)$.

Lemma 3. *(Lemma 6.6 in Garber and Hazan [2016]) Let $\{\ell_t(\mathbf{x})\}_{t=1}^T$ be a sequence of functions over a set $\mathcal{K}$, and let $\mathbf{x}_t^* \in \arg\min_{\mathbf{x} \in \mathcal{K}} \sum_{i=1}^t \ell_i(\mathbf{x})$ for any $t \in [T]$. Then, it holds that $\sum_{t=1}^T \ell_t(\mathbf{x}_t^*) - \min_{\mathbf{x} \in \mathcal{K}} \sum_{t=1}^T \ell_t(\mathbf{x}) \leq 0$.*

Specifically, by applying Lemma 3, we have $\sum_{m=1}^{T/K} \ell_m(\mathbf{y}_{m+1}^*) - \sum_{m=1}^{T/K} \ell_m(\tilde{\mathbf{x}}^*) \leq 0$. Combining this inequality with Assumption 1, we have

$$\sum_{m=1}^{T/K} \langle \nabla_m, \mathbf{y}_{m+1}^* - \tilde{\mathbf{x}}^* \rangle \leq \frac{\|\tilde{\mathbf{x}}^* - \mathbf{y}_1\|_2^2}{\eta} - \frac{\|\mathbf{y}_2^* - \mathbf{y}_1\|_2^2}{\eta} \leq \frac{4R^2}{\eta}. \quad (15)$$

Moreover, to replace $\mathbf{y}_{m+1}^*$ in the left side of (15) with $\mathbf{y}_m^*$, we introduce the following lemma.

Lemma 4. (Lemma 5 in Duchi et al. [2011]) Let $\Pi_{\mathcal{K}}(\mathbf{u}, \eta) = \operatorname{argmin}_{\mathbf{x} \in \mathcal{K}} \left\{ \langle \mathbf{u}, \mathbf{x} \rangle + \frac{1}{\eta} \|\mathbf{x}\|_2^2 \right\}$. We have $\|\Pi_{\mathcal{K}}(\mathbf{u}, \eta) - \Pi_{\mathcal{K}}(\mathbf{v}, \eta)\|_2 \leq \frac{\eta}{2} \|\mathbf{u} - \mathbf{v}\|_2$.

Combining Lemma 4 with (5) for convex functions, we have

$$\|\mathbf{y}_m^* - \mathbf{y}_{m+1}^*\|_2 \leq \frac{\eta}{2} \left\| \left(\sum_{i=1}^{m-1} \nabla_i - \frac{2\mathbf{y}_1}{\eta} \right) - \left(\sum_{i=1}^m \nabla_i - \frac{2\mathbf{y}_1}{\eta} \right) \right\|_2 = \frac{\eta}{2} \|\nabla_m\|_2. \quad (16)$$

Then, combining (15) with (16), we have

$$\begin{aligned} \sum_{m=1}^{T/K} \langle \nabla_m, \mathbf{y}_m^* - \tilde{\mathbf{x}}^* \rangle &= \sum_{m=1}^{T/K} \langle \nabla_m, \mathbf{y}_{m+1}^* - \tilde{\mathbf{x}}^* + \mathbf{y}_m^* - \mathbf{y}_{m+1}^* \rangle \\ &\leq \frac{4R^2}{\eta} + \sum_{m=1}^{T/K} \|\nabla_m\|_2 \|\mathbf{y}_m^* - \mathbf{y}_{m+1}^*\|_2 \leq \frac{4R^2}{\eta} + \frac{\eta}{2} \sum_{m=1}^{T/K} \|\nabla_m\|_2^2. \end{aligned} \quad (17)$$

We notice that the term $\|\nabla_m\|_2^2$ in (17) can directly benefit from the blocking update mechanism, as shown by the upper bound in the following lemma.

Lemma 5. Under Assumptions 2 and 3, for any $m \in [T/K]$, Algorithm 1 ensures $\mathbb{E}[\|\nabla_m\|_2^2] \leq K \left(\frac{nM}{\delta} \right)^2 + K^2 G^2$.

However, to completely bound the right side of (14), we still need to analyze $\|\mathbf{y}_m - \mathbf{y}_m^*\|_2$, which is more complicated due to the effect of delays. Specifically, let

$$\mathcal{U}_m = \{1, \dots, (m-1)K\} \setminus \cup_{t=1}^{(m-1)K} \mathcal{F}_t \quad (18)$$

be the set consisting of the time stamp of loss values that are queried but still not arrive at the end of round $(m-1)K$. By using Lemma 4 again, we have

$$\|\mathbf{y}_m - \mathbf{y}_m^*\|_2 \leq \frac{\eta}{2} \left\| \left(\bar{\mathbf{g}}_{(m-1)K} - \frac{2\mathbf{y}_1}{\eta} \right) - \left(\sum_{i=1}^{m-1} \nabla_i - \frac{2\mathbf{y}_1}{\eta} \right) \right\|_2 = \frac{\eta}{2} \left\| \sum_{t \in \mathcal{U}_m} \frac{n}{\delta} f_t(\mathbf{x}_t) \mathbf{u}_t \right\|_2. \quad (19)$$

Moreover, we establish the following lemma regarding the right side of (19).

Lemma 6. Under Assumptions 2 and 3, for any $m \in [T/K]$, Algorithm 1 ensures

$$\mathbb{E} \left[\left\| \sum_{t \in \mathcal{U}_m} \frac{n}{\delta} f_t(\mathbf{x}_t) \mathbf{u}_t \right\|_2^2 \right] \leq 2 \left(\frac{d^2}{K^2} + 4 \right) \left(K \left(\frac{nM}{\delta} \right)^2 + K^2 G^2 \right).$$

Combining (14), (17), (19), Lemma 5, Lemma 6, and $\gamma = K \left(\frac{nM}{\delta} \right)^2 + K^2 G^2$, we have

$$\begin{aligned} \mathbb{E}[\operatorname{Reg}(T)] &\leq \frac{4R^2}{\eta} + \mathbb{E} \left[\frac{\eta}{2} \sum_{m=1}^{T/K} \|\nabla_m\|_2^2 \right] + KG \sum_{m=1}^{T/K} \mathbb{E}[\|\mathbf{y}_m - \mathbf{y}_m^*\|_2] + 3\delta GT + \frac{\delta GRT}{r} \\ &\leq \frac{4R^2}{\eta} + \frac{\eta T \gamma}{2K} + \frac{\eta TG}{2} \sqrt{2 \left(\frac{d^2}{K^2} + 4 \right) \gamma} + 3\delta GT + \frac{\delta GRT}{r}. \end{aligned}$$

4 Experiments

In this section, we compare our D-FTBL against GOLD [Héliou et al., 2020] and improved GOLD [Bistritz et al., 2022] by conducting simulation experiments on two publicly available data sets—ijcnn1 and SUSY from the LIBSVM repository [Chang and Lin, 2011]. All algorithms are implemented with Python, and tested on a laptop with 2.4GHz CPU and 16GB memory.

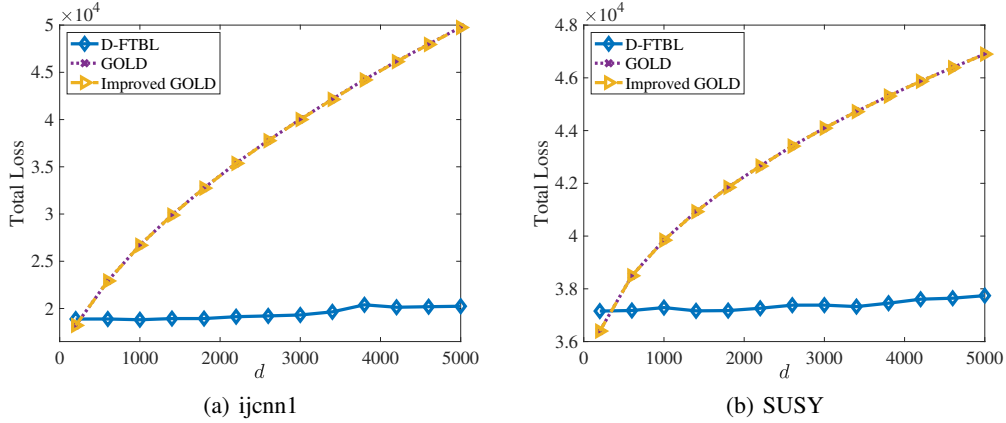


Figure 1: Experimental results on delayed online binary classification for ijcn1 and SUSY.

Specifically, we randomly select $T = 40000$ examples from the original data sets, and consider online binary classification over a convex set $\mathcal{K} = \{\mathbf{x} \in \mathbb{R}^n \mid \|\mathbf{x}\|_2 \leq 50\}$. The dimensionality of ijcn1 and SUSY are $n = 22$ and $n = 18$, respectively. In each round $t \in [T]$, the adversary chooses the hinge loss

$$f_t(\mathbf{x}) = \max\{1 - y_t \mathbf{w}_t^\top \mathbf{x}, 0\}$$

where $\mathbf{w}_t$ and $y_t \in \{-1, 1\}$ are the feature vector and class label of the t -th example, respectively. Different values of the maximum delay d in the set $\{200, 600, 1000, \dots, 5000\}$ have been tried in our experiments. For each specific d , to simulate arbitrary delays, d_t is independently and uniformly sampled from $[d]$. In this way, the average delay $\bar{d}$ is equal to $(d + 1)/2$ in expectation, and thus is close to the maximum delay.

According to the previous discussions about Theorem 1, we set $\alpha = 0$, $K = \lfloor n\sqrt{T} \rfloor$, $\delta = c\sqrt{n}T^{-1/4}$, and $\eta = c' / \max\{\sqrt{Td}, \sqrt{n}T^{3/4}\}$ for our D-FTBL by tuning these two constants c and c' . For those two baselines, we only need to set parameters δ and η . In addition to the theoretically suggested value of δ and η , we also introduce c and c' as the scale factor, respectively. For all algorithms, c and c' are respectively selected from $\{0.1, 1.0, 10\}$ and $\{0.01, 0.1, \dots, 100\}$ simply according to their performance for $d = 200$. Moreover, due to the randomness of these algorithms, we repeat them 20 times and report the average of their total loss.

Fig. 1 shows the results of all algorithms on both data sets. We first find that when d increases from 200 to 5000, the total loss of our D-FTBL grows slowly, which is consistent with the dependence of our regret bound on d . It is worth noting that $d = 5000$ is larger than $n\sqrt{T}$ in our experiments. Second, from $d = 600$ to $d = 5000$, the total loss of our D-FTBL is better than both GOLD and improved GOLD, which verifies the advantage of our algorithm in the delayed setting. By contrast, due to $\bar{d} \approx d$, the performance of improved GOLD is very close to that of GOLD. Finally, we also notice that D-FTBL is slightly worse than baselines for $d = 200$. However, it is reasonable because the block update mechanism enlarges each delay to be at least the block size, which could result in a slightly larger constant factor in the regret.

5 Conclusion and future work

In this paper, we investigate BCO with delayed feedback, and propose a novel algorithm called D-FTBL by exploiting the blocking update mechanism. Our analysis first reveals that it can achieve a regret bound of $O(\sqrt{n}T^{3/4} + \sqrt{dT})$ in general, which improves the delay-dependent part of the existing $O(\sqrt{n}T^{3/4} + (n\bar{d})^{1/3}T^{2/3})$ regret bound as long as d is not larger than $O((n\bar{d})^{2/3}T^{1/3})$. Furthermore, we consider the special case with strongly convex functions, and prove that the regret of D-FTBL can be reduced to $O((nT)^{2/3} \log^{1/3} T + d \log T)$. Finally, if the action sets are unconstrained, we show that D-FTBL can be simply extended to enjoy the $O(n\sqrt{T} \log T + d \log T)$ regret for strongly convex and smooth functions. Nonetheless, there still exist several open problems, which are discussed in the appendix due to the limitation of space.

Acknowledgments

This work was partially supported by the National Natural Science Foundation of China (62306275, U23A20382), the Zhejiang Province High-Level Talents Special Support Program “Leading Talent of Technological Innovation of TenThousands Talents Program” (No. 2022R52046), the Key Research and Development Program of Zhejiang Province (No. 2023C03192), and the Open Research Fund of the State Key Laboratory of Blockchain and Data Security, Zhejiang University. The authors would also like to thank Chenxu Zhang for helping conduct experiments.

References

- Jacob D. Abernethy, Peter L. Bartlett, Alexander Rakhlin, and Ambuj Tewari. Optimal strategies and minimax lower bounds for online convex games. In *Proceedings of the 21st Annual Conference on Learning Theory*, pages 415–424, 2008.
- Alekh Agarwal and John C. Duchi. Distributed delayed stochastic optimization. In *Advances in Neural Information Processing Systems 24*, pages 873–881, 2011.
- Alekh Agarwal, Ofer Dekel, and Lin Xiao. Optimal algorithms for online convex optimization with multi-point bandit feedback. In *Proceedings of the 23rd Annual Conference on Learning Theory*, pages 28–40, 2010.
- Baruch Awerbuch and Robert Kleinberg. Online linear optimization and adaptive routing. *Journal of Computer and System Sciences*, 74(1):97–114, 2008.
- Francis Bach and Vianney Perchet. Highly-smooth zero-th order online optimization. In *Proceedings of the 29th Annual Conference on Learning Theory*, pages 257–283, 2016.
- Ilai Bistrutz, Zhengyuan Zhou, Xi Chen, Nicholas Bambos, and Jose Blanchet. No weighted-regret learning in adversarial bandits with delays. *Journal of Machine Learning Research*, 23(139):1–43, 2022.
- Stephen Boyd and Lieven Vandenbergh. *Convex Optimization*. Cambridge University Press, 2004.
- Sébastien Bubeck and Ronen Eldan. Multi-scale exploration of convex functions and bandit convex optimization. In *Proceedings of the 29th Annual Conference on Learning Theory*, pages 583–589, 2016.
- Sébastien Bubeck, Yin Tat Lee, and Ronen Eldan. Kernel-based methods for bandit convex optimization. In *Proceedings of the 49th Annual ACM SIGACT Symposium on Theory of Computing*, pages 72–85, 2017.
- Sébastien Bubeck, Ronen Eldan, and Yin Tat Lee. Kernel-based methods for bandit convex optimization. *Journal of the ACM*, 68(4):1–25, 2021.
- Nicolò Cesa-Bianchi, Yoav Freund, David Haussler, David P. Helmbold, Robert E. Schapire, and Manfred K. Warmuth. How to use expert advice. *Journal of the ACM*, 44(3):427–485, 1997.
- Chih-Chung Chang and Chih-Jen Lin. LIBSVM: A library for support vector machines. *ACM Transactions on Intelligent Systems and Technology*, 2(27):1–27, 2011.
- Sijia Chen, Yu-Jie Zhang, Wei-Wei Tu, Peng Zhao, and Lijun Zhang. Optimistic online mirror descent for bridging stochastic and adversarial online convex optimization. *Journal of Machine Learning Research*, 25(178):1–62, 2024.
- John C. Duchi, Alekh Agarwal, and Martin J. Wainwright. Dual averaging for distributed optimization: Convergence analysis and network scaling. *IEEE Transactions on Automatic Control*, 57(3):592–606, 2011.
- John C. Duchi, Michael I. Jordan, Martin J. Wainwright, and Andre Wibisono. Optimal rates for zero-order convex optimization: The power of two function evaluations. *IEEE Transactions on Information Theory*, 61(5):2788–2806, 2015.

- Genevieve E Flaspohler, Francesco Orabona, Judah Cohen, Soukayna Mouatadid, Miruna Oprescu, Paulo Orenstein, and Lester Mackey. Online learning with optimism and delay. In *Proceedings of the 38th International Conference on Machine Learning*, pages 3363–3373, 2021.
- Abraham D. Flaxman, Adam Tauman Kalai, and H. Brendan McMahan. Online convex optimization in the bandit setting: Gradient descent without a gradient. In *Proceedings of the 16th Annual ACM-SIAM Symposium on Discrete Algorithms*, pages 385–394, 2005.
- Dan Garber and Elad Hazan. A linearly convergent conditional gradient algorithm with applications to online and stochastic optimization. *SIAM Journal on Optimization*, 26(3):1493–1528, 2016.
- Dan Garber and Ben Kretzu. Improved regret bounds for projection-free bandit convex optimization. In *Proceedings of the 23rd International Conference on Artificial Intelligence and Statistics*, pages 2196–2206, 2020.
- Dan Garber and Ben Kretzu. Revisiting projection-free online learning: the Strongly convex case. In *Proceedings of the 24th International Conference on Artificial Intelligence and Statistics*, pages 3592–3600, 2021.
- Paula Gradu, John Hallman, and Elad Hazan. Non-stochastic control with bandit feedback. In *Advances in Neural Information Processing Systems 33*, pages 10764–10774, 2020.
- Elad Hazan. Introduction to online convex optimization. *Foundations and Trends in Optimization*, 2(3–4):157–325, 2016.
- Elad Hazan and Satyen Kale. Projection-free online learning. In *Proceedings of the 29th International Conference on Machine Learning*, pages 1843–1850, 2012.
- Elad Hazan and Kfir Y. Levy. Bandit convex optimization: Towards tight bounds. In *Advances in Neural Information Processing Systems 27*, pages 784–792, 2014.
- Elad Hazan and Yuanzhi Li. An optimal algorithm for bandit convex optimization. *arXiv:1603.04350*, 2016.
- Elad Hazan and Edgar Minasyan. Faster projection-free online learning. In *Proceedings of the 33rd Annual Conference on Learning Theory*, pages 1877–1893, 2020.
- Elad Hazan, Amit Agarwal, and Satyen Kale. Logarithmic regret algorithms for online convex optimization. *Machine Learning*, 69(2):169–192, 2007.
- Amélie Héliou, Panayotis Mertikopoulos, and Zhengyuan Zhou. Gradient-free online learning in games with delayed rewards. In *Proceedings of the 37th International Conference on Machine Learning*, pages 4172–4181, 2020.
- Shinji Ito. An optimal algorithm for bandit convex optimization with strongly-convex and smooth loss. In *Proceedings of the 23rd International Conference on Artificial Intelligence and Statistics*, pages 2229–2239, 2020.
- Pooria Joulani, András György, and Csaba Szepesvári. Online learning under delayed feedback. In *Proceedings of the 30th International Conference on Machine Learning*, pages 1453–1461, 2013.
- Pooria Joulani, András György, and Csaba Szepesvári. Delay-tolerant online convex optimization: Unified analysis and adaptive-gradient algorithms. *Proceedings of the 30th AAAI Conference on Artificial Intelligence*, pages 1744–1750, 2016.
- John Langford, Alexander J. Smola, and Martin Zinkevich. Slow learners are fast. In *Advances in Neural Information Processing Systems 22*, pages 2331–2339, 2009.
- Tor Lattimore. Improved regret for zeroth-order adversarial bandit convex optimisation. *Mathematical Statistics and Learning*, 2(3/4):311–334, 2020.
- Tor Lattimore. Bandit convex optimisation. *arXiv:2402.06535*, 2024.

- Bingcong Li, Tianyi Chen, and Georgios B. Giannakis. Bandit online learning with unknown delays. In *Proceedings of the 22nd International Conference on Artificial Intelligence and Statistics*, pages 993–1002, 2019.
- Sadhika Malladi, Tianyu Gao, Eshaan Nichani, Alex Damian, Jason D. Lee, Danqi Chen, and Sanjeev Arora. Fine-tuning language models with just forward passes. In *Advances in Neural Information Processing Systems 36*, pages 53038–53075, 2023.
- H. Brendan McMahan and Matthew Streeter. Delay-tolerant algorithms for asynchronous distributed online learning. In *Advances in Neural Information Processing Systems 27*, pages 2915–2923, 2014.
- H. Brendan McMahan, Gary Holt, D. Sculley, Michael Young, Dietmar Ebner, Julian Grady, Lan Nie, Todd Phillips, Eugene Davydov, Daniel Golovin, Sharat Chikkerur, Dan Liu, Martin Wattenberg, Arnar Mar Hrafnkelsson, Tom Boulos, and Jeremy Kubica. Ad click prediction: A view from the trenches. In *Proceedings of the 19th ACM SIGKDD International Conference on Knowledge Discovery and Data Mining*, pages 1222–1230, 2013.
- Arkadi Nemirovski. Interior point polynomial time methods in convex programming. Lecture notes, Technion – Israel Institute of Technology, 2004.
- Kent Quanrud and Daniel Khashabi. Online learning with adversarial delays. In *Advances in Neural Information Processing Systems 28*, pages 1270–1278, 2015.
- Ankan Saha and Ambuj Tewari. Improved regret guarantees for online smooth convex optimization with bandit feedback. In *Proceedings of the 14th International Conference on Artificial Intelligence and Statistics*, pages 636–642, 2011.
- Shai Shalev-Shwartz. Online learning and online convex optimization. *Foundations and Trends in Machine Learning*, 4(2):107–194, 2011.
- Ohad Shamir. On the complexity of bandit and derivative-free stochastic convex optimization. In *Proceedings of the 26th Conference on Learning Theory*, pages 3–24, 2013.
- Ohad Shamir. An optimal algorithm for bandit and zero-order convex optimization with two-point feedback. *Journal of Machine Learning Research*, 18(52):1–11, 2017.
- Y. Jennifer Sun, Stephen Newman, and Elad Hazan. Optimal rates for bandit nonstochastic control. In *Advances in Neural Information Processing Systems 36*, pages 21908–21919, 2023.
- Dirk van der Hoeven, Ashok Cutkosky, and Haipeng Luo. Comparator-adaptive convex bandits. In *Advances in Neural Information Processing Systems 33*, pages 19795–19804, 2020.
- Yuanyu Wan, Wei-Wei Tu, and Lijun Zhang. Projection-free distributed online convex optimization with $O(\sqrt{T})$ communication complexity. In *Proceedings of the 37th International Conference on Machine Learning*, pages 9818–9828, 2020.
- Yuanyu Wan, Wei-Wei Tu, and Lijun Zhang. Online strongly convex optimization with unknown delays. *Machine Learning*, 111(3):871–893, 2022a.
- Yuanyu Wan, Wei-Wei Tu, and Lijun Zhang. Online Frank-Wolfe with arbitrary delays. In *Advances in Neural Information Processing Systems 35*, pages 19703–19715, 2022b.
- Yuanyu Wan, Guanghui Wang, Wei-Wei Tu, and Lijun Zhang. Projection-free distributed online learning with sublinear communication complexity. *Journal of Machine Learning Research*, 23(172):1–53, 2022c.
- Yuanyu Wan, Yibo Wang, Chang Yao, Wei-Wei Tu, and Lijun Zhang. Projection-free online learning with arbitrary delays. *arXiv:2204.04964v2*, 2023.
- Yuanyu Wan, Chang Yao, Mingli Song, and Lijun Zhang. Non-stationary online convex optimization with arbitrary delays. In *Proceedings of the 41st International Conference on Machine Learning*, pages 49991–50011, 2024.

- Yibo Wang, Yuanyu Wan, Shimao Zhang, and Lijun Zhang. Distributed projection-free online learning for smooth and convex losses. In *Proceedings of the 37th AAAI Conference on Artificial Intelligence*, pages 10226–10234, 2023.
- Yibo Wang, Sijia Chen, Wei Jiang, Wenhao Yang, Yuanyu Wan, and Lijun Zhang. Online composite optimization between stochastic and adversarial environments. In *Advances in Neural Information Processing Systems 37*, 2024a.
- Yibo Wang, Wenhao Yang, Wei Jiang, Shiyin Lu, Bing Wang, Haihong Tang, Yuanyu Wan, and Lijun Zhang. Non-stationary projection-free online learning with dynamic and adaptive regret guarantees. In *Proceedings of the 38th AAAI Conference on Artificial Intelligence*, pages 15671–15679, 2024b.
- Marcelo J. Weinberger and Erik Ordentlich. On delayed prediction of individual sequences. *IEEE Transactions on Information Theory*, 48(7):1959–1976, 2002.
- Mingrui Zhang, Lin Chen, Hamed Hassani, and Amin Karbasi. Online continuous submodular maximization: From full-information to bandit feedback. In *Advances in Neural Information Processing Systems 32*, pages 9210–9221, 2019.
- Yihua Zhang, Pingzhi Li, Junyuan Hong, Jiayang Li, Yimeng Zhang, Wenqing Zheng, Pin-Yu Chen, Jason D. Lee, Wotao Yin, Mingyi Hong, Zhangyang Wang, Sijia Liu, and Tianlong Chen. Revisiting zeroth-order optimization for memory-efficient LLM fine-tuning: A benchmark. In *Proceedings of the 41st International Conference on Machine Learning*, pages 59173–59190, 2024.
- Martin Zinkevich. Online convex programming and generalized infinitesimal gradient ascent. In *Proceedings of the 20th International Conference on Machine Learning*, pages 928–936, 2003.

A Detailed discussions on future work

First, we notice that all our regret bounds depend on the maximum delay, and thus it is natural to investigate whether these bounds can be further improved to be depending on the average delay. It seems highly non-trivial to obtain such results with our D-FTBL because the blocking update mechanism actually enlarges each delay to be at least the block size.

Second, it is appealing to apply our algorithm to an emerging task—memory-efficient fine-tuning of large language models [Malladi et al., 2023, Zhang et al., 2024]. The key insight is that our algorithm only requires the delayed loss value to update the model, and thus could achieve an asynchronous acceleration while avoiding memory costs of the backpropagation.

Moreover, in the above task, we actually only need to handle a sequence of stochastic loss functions. Note that Agarwal and Duchi [2011] have shown that the delay only increases the regret of OCO with stochastic and smooth loss functions in an additive way, i.e., an $O(\sqrt{T} + d^2)$ regret bound. Thus, it is interesting to investigate whether the stochastic setting or an intermediate setting [Chen et al., 2024, Wang et al., 2024a] can make delayed BCO easier in a similar way.

Finally, it is also worth extending other BCO algorithms into the delayed setting, e.g., generalizing the algorithm of Saha and Tewari [2011] to improve the regret for smooth functions. However, a more complicated analysis is required because they utilize additional techniques, e.g., the self-concordant barrier [Nemirovski, 2004].

B Proof of Lemma 2

Recall the definition of $\mathbf{x}^*$ and $\tilde{\mathbf{x}}^*$ in the proof of Theorem 1. First, it is easy to verify that

$$\begin{aligned} \text{Reg}(T) &= \sum_{m=1}^{T/K} \sum_{t=(m-1)K+1}^{mK} (f_t(\mathbf{y}_m + \delta \mathbf{u}_t) - f_t(\mathbf{x}^*)) \\ &\leq \sum_{m=1}^{T/K} \sum_{t=(m-1)K+1}^{mK} \left(f_t(\mathbf{y}_m) + G\|\delta \mathbf{u}_t\|_2 + f_t(\tilde{\mathbf{x}}^*) - G \left\| \frac{\delta}{r} \mathbf{x}^* \right\|_2 \right) \\ &\leq \sum_{m=1}^{T/K} \sum_{t=(m-1)K+1}^{mK} (f_t(\mathbf{y}_m) - f_t(\tilde{\mathbf{x}}^*)) + \delta GT + \frac{\delta GRT}{r} \end{aligned} \quad (20)$$

where the first inequality is due to Assumption 2, and the second inequality is due to Assumption 1 and $\mathbf{u}_t \sim \mathcal{S}^n$. Note that $\mathbf{y}_1, \dots, \mathbf{y}_{T/K}$ in Algorithm 1 are computed according to approximate gradients of the δ -smoothed version of original functions, i.e., $\hat{f}_{t,\delta}(\mathbf{x}) = \mathbb{E}_{\mathbf{u} \sim \mathcal{B}^n} [f_t(\mathbf{x} + \delta \mathbf{u})]$, $\forall t \in [T]$. Therefore, before utilizing the unbiasedness of these approximate gradients in Lemma 1, we introduce the following lemma regarding the connection between the original function and its δ -smoothed version over $\mathcal{K}_\delta$ defined in (3).

Lemma 7. (Lemma 2.6 in Hazan [2016]) Let $f(\mathbf{x}) : \mathbb{R}^n \rightarrow \mathbb{R}$ be α -strongly convex and G -Lipschitz over a set $\mathcal{K}$ satisfying Assumption 1. Its δ -smoothed version $\hat{f}_\delta(\mathbf{x})$ defined in (2) has the following properties:

- $\hat{f}_\delta(\mathbf{x})$ is α -strongly convex over $\mathcal{K}_\delta$;
- $|\hat{f}_\delta(\mathbf{x}) - f(\mathbf{x})| \leq \delta G$ for any $\mathbf{x} \in \mathcal{K}_\delta$;
- $\hat{f}_\delta(\mathbf{x})$ is G -Lipschitz over $\mathcal{K}_\delta$.

Combining (20) with the second property in Lemma 7, it is easy to verify that

$$\text{Reg}(T) \leq \sum_{m=1}^{T/K} \sum_{t=(m-1)K+1}^{mK} \left(\hat{f}_{t,\delta}(\mathbf{y}_m) - \hat{f}_{t,\delta}(\tilde{\mathbf{x}}^*) + 2\delta G \right) + \delta GT + \frac{\delta GRT}{r}. \quad (21)$$

Then, combining (21) with the first property in Lemma 7 where $\alpha = 0$, we have

$$\text{Reg}(T) \leq \sum_{m=1}^{T/K} \sum_{t=(m-1)K+1}^{mK} \langle \nabla \hat{f}_{t,\delta}(\mathbf{y}_m), \mathbf{y}_m - \tilde{\mathbf{x}}^* \rangle + 3\delta GT + \frac{\delta GRT}{r}.$$

From the above inequality, it is easy to verify that

$$\begin{aligned} \text{Reg}(T) &\leq \sum_{m=1}^{T/K} \sum_{t=(m-1)K+1}^{mK} \langle \nabla \hat{f}_{t,\delta}(\mathbf{y}_m), \mathbf{y}_m^* - \tilde{\mathbf{x}}^* + \mathbf{y}_m - \mathbf{y}_m^* \rangle + 3\delta GT + \frac{\delta GRT}{r} \\ &\leq \sum_{m=1}^{T/K} \sum_{t=(m-1)K+1}^{mK} \langle \nabla \hat{f}_{t,\delta}(\mathbf{y}_m), \mathbf{y}_m^* - \tilde{\mathbf{x}}^* \rangle + KG \sum_{m=1}^{T/K} \|\mathbf{y}_m - \mathbf{y}_m^*\|_2 + 3\delta GT + \frac{\delta GRT}{r} \end{aligned} \quad (22)$$

where the last inequality is due to the last property in Lemma 7.

Moreover, according to Lemma 1, we have

$$\begin{aligned} \sum_{t=(m-1)K+1}^{mK} \mathbb{E} \left[\langle \nabla \hat{f}_{t,\delta}(\mathbf{y}_m), \mathbf{y}_m^* - \tilde{\mathbf{x}}^* \rangle \right] &= \sum_{t=(m-1)K+1}^{mK} \mathbb{E} \left[\left\langle \frac{n}{\delta} f_t(\mathbf{y}_m + \delta \mathbf{u}_t) \mathbf{u}_t, \mathbf{y}_m^* - \tilde{\mathbf{x}}^* \right\rangle \right] \\ &= \mathbb{E} [\langle \nabla_m, \mathbf{y}_m^* - \tilde{\mathbf{x}}^* \rangle]. \end{aligned}$$

Finally, we can complete this proof by first taking expectations of both sides in (22) and then substituting the above equality into the right side.

C Proof of Lemmas 5 and 6

Lemma 5 can be proved by simply following the proof of Lemma 5 in Garber and Kretzu [2020]. In the following, we first prove Lemma 6, and then include a simple proof of Lemma 5 for completeness.

For brevity, let $\mathbf{g}_t = \frac{n}{\delta} f_t(\mathbf{x}_t) \mathbf{u}_t$ for any $t \in [T]$. Since $\mathbf{g}_1, \dots, \mathbf{g}_{(m-1)K-d+1}$ must be available at the end of round $(m-1)K$, it is not hard to verify that

$$\left\| \sum_{t \in \mathcal{U}_m} \mathbf{g}_t \right\|_2^2 = \left\| \sum_{k=m-1-\lceil d/K \rceil}^{m-1} \sum_{t \in \mathcal{A}_k} \mathbf{g}_t \right\|_2^2 \leq \left(\left\lceil \frac{d}{K} \right\rceil + 1 \right) \sum_{k=m-1-\lceil d/K \rceil}^{m-1} \left\| \sum_{t \in \mathcal{A}_k} \mathbf{g}_t \right\|_2^2 \quad (23)$$

where $\mathcal{A}_k = \{(k-1)K+1, \dots, kK\} \cap \mathcal{U}_m$.

Because of $|\mathcal{A}_k| \leq K$, for any $k = m-1-\lceil d/K \rceil, \dots, m-1$, we have

$$\begin{aligned} \mathbb{E} \left[\left\| \sum_{t \in \mathcal{A}_k} \mathbf{g}_t \right\|_2^2 \right] &= \mathbb{E} \left[\sum_{t \in \mathcal{A}_k} \|\mathbf{g}_t\|_2^2 + \sum_{i,j \in \mathcal{A}_k, i \neq j} \langle \mathbf{g}_i, \mathbf{g}_j \rangle \right] \\ &\leq |\mathcal{A}_k| \left(\frac{nM}{\delta} \right)^2 + \mathbb{E} \left[\sum_{i,j \in \mathcal{A}_k, i \neq j} \langle \mathbb{E}[\mathbf{g}_i | \mathbf{y}_k], \mathbb{E}[\mathbf{g}_j | \mathbf{y}_k] \rangle \right] \\ &\leq K \left(\frac{nM}{\delta} \right)^2 + \mathbb{E} \left[\sum_{i,j \in \mathcal{A}_k, i \neq j} \|\mathbb{E}[\mathbf{g}_i | \mathbf{y}_k]\|_2 \|\mathbb{E}[\mathbf{g}_j | \mathbf{y}_k]\|_2 \right] \\ &\leq K \left(\frac{nM}{\delta} \right)^2 + (|\mathcal{A}_k|^2 - |\mathcal{A}_k|) G^2 \leq K \left(\frac{nM}{\delta} \right)^2 + K^2 G^2 \end{aligned} \quad (24)$$

where the first inequality is due to Assumption 3, and the third inequality is due to Assumption 2, Lemma 1, and the last property in Lemma 7.

Combining (23) with (24), we have

$$\mathbb{E} \left[\left\| \sum_{t \in \mathcal{U}_m} \mathbf{g}_t \right\|_2^2 \right] \leq 2 \left(\frac{d^2}{K^2} + 4 \right) \left(K \left(\frac{nM}{\delta} \right)^2 + K^2 G^2 \right)$$

which completes the proof of Lemma 6.

Additionally, let $\mathcal{T}_m = \{(m-1)K+1, \dots, mK\}$. Following (24), it is easy to verify that

$$\mathbb{E} \left[\|\nabla_m\|_2^2 \right] = \mathbb{E} \left[\sum_{t \in \mathcal{T}_m} \|\mathbf{g}_t\|_2^2 + \sum_{i,j \in \mathcal{T}_m, i \neq j} \langle \mathbf{g}_i, \mathbf{g}_j \rangle \right] \leq K \left(\frac{nM}{\delta} \right)^2 + K^2 G^2 \quad (25)$$

which completes the proof of Lemma 5.

D Proof of Theorem 2

This proof is similar to that of Theorem 1, but requires some specific extensions to utilize the strong convexity. Note that by combining (21) in the proof of Lemma 2 with the strong convexity, we have

$$\text{Reg}(T) \leq \sum_{m=1}^{T/K} \sum_{t \in \mathcal{T}_m} \left(\langle \nabla \hat{f}_{t,\delta}(\mathbf{y}_m), \mathbf{y}_m - \tilde{\mathbf{x}}^* \rangle - \frac{\alpha}{2} \|\mathbf{y}_m - \tilde{\mathbf{x}}^*\|_2^2 \right) + 3\delta GT + \frac{\delta GRT}{r} \quad (26)$$

where $\mathcal{T}_m = \{(m-1)K+1, \dots, mK\}$, $\tilde{\mathbf{x}}^* = (1-\delta/r)\mathbf{x}^*$, and $\mathbf{x}^* \in \text{argmin}_{\mathbf{x} \in \mathcal{K}} \sum_{t=1}^T f_t(\mathbf{x})$.

Moreover, by reorganizing (26) and inserting the ideal action $\mathbf{y}_m^*$,⁴ we have

$$\begin{aligned} & \text{Reg}(T) - \left(3\delta GT + \frac{\delta GRT}{r} \right) \\ & \leq \sum_{m=1}^{T/K} \sum_{t \in \mathcal{T}_m} \left(\langle \nabla \hat{f}_{t,\delta}(\mathbf{y}_m), \mathbf{y}_m^* - \tilde{\mathbf{x}}^* \rangle + \langle \nabla \hat{f}_{t,\delta}(\mathbf{y}_m), \mathbf{y}_m - \mathbf{y}_m^* \rangle - \frac{\alpha}{2} \|\mathbf{y}_m - \tilde{\mathbf{x}}^*\|_2^2 \right) \\ & \leq \sum_{m=1}^{T/K} \sum_{t \in \mathcal{T}_m} \left(\langle \nabla \hat{f}_{t,\delta}(\mathbf{y}_m), \mathbf{y}_m^* - \tilde{\mathbf{x}}^* \rangle - \frac{\alpha}{2} \|\mathbf{y}_m - \tilde{\mathbf{x}}^*\|_2^2 \right) + \sum_{m=1}^{T/K} KG \|\mathbf{y}_m - \mathbf{y}_m^*\|_2 \end{aligned} \quad (27)$$

where the last equality is due to Assumption 2 and the last property in Lemma 7.

For brevity, we define surrogate losses as $\ell_m(\mathbf{x}) = \langle \nabla_m, \mathbf{x} \rangle + \frac{\alpha K}{2} \|\mathbf{y}_m - \mathbf{x}\|_2^2$ for $m \in [T/K]$, and notice that

$$\mathbf{y}_{m+1}^* = \text{argmin}_{\mathbf{x} \in \mathcal{K}_\delta} \sum_{i=1}^m \ell_i(\mathbf{x}) \quad (28)$$

in the strongly convex case. Then, it is not hard to verify that

$$\begin{aligned} & \mathbb{E} \left[\sum_{m=1}^{T/K} \sum_{t \in \mathcal{T}_m} \left(\langle \nabla \hat{f}_{t,\delta}(\mathbf{y}_m), \mathbf{y}_m^* - \tilde{\mathbf{x}}^* \rangle - \frac{\alpha}{2} \|\mathbf{y}_m - \tilde{\mathbf{x}}^*\|_2^2 \right) \right] \\ & = \mathbb{E} \left[\sum_{m=1}^{T/K} \sum_{t \in \mathcal{T}_m} \left(\left\langle \frac{n}{\delta} f_t(\mathbf{y}_m + \delta \mathbf{u}_t) \mathbf{u}_t, \mathbf{y}_m^* - \tilde{\mathbf{x}}^* \right\rangle - \frac{\alpha}{2} \|\mathbf{y}_m - \tilde{\mathbf{x}}^*\|_2^2 \right) \right] \\ & = \mathbb{E} \left[\sum_{m=1}^{T/K} \left(\langle \nabla_m, \mathbf{y}_{m+1}^* - \tilde{\mathbf{x}}^* + \mathbf{y}_m^* - \mathbf{y}_{m+1}^* \rangle - \frac{\alpha K}{2} \|\mathbf{y}_m - \tilde{\mathbf{x}}^*\|_2^2 \right) \right] \\ & \leq \mathbb{E} \left[\sum_{m=1}^{T/K} (\ell_m(\mathbf{y}_{m+1}^*) - \ell_m(\tilde{\mathbf{x}}^*)) \right] + \mathbb{E} \left[\sum_{m=1}^{T/K} \|\nabla_m\|_2 \|\mathbf{y}_m^* - \mathbf{y}_{m+1}^*\|_2 \right] \\ & \leq \mathbb{E} \left[\sum_{m=1}^{T/K} \|\nabla_m\|_2 \|\mathbf{y}_m^* - \mathbf{y}_{m+1}^*\|_2 \right] \end{aligned} \quad (29)$$

where the first equality is due to Lemma 1, and the last inequality is due to (28) and Lemma 3. From (27) and (29), we still need to bound $\|\mathbf{y}_m - \mathbf{y}_m^*\|_2$ and $\|\mathbf{y}_m^* - \mathbf{y}_{m+1}^*\|_2$.

⁴Note that the definition of $\mathbf{y}_m^*$ in (5) for strongly convex functions is only valid for $m \geq 2$. For $m = 1$, we simply set $\mathbf{y}_1^* = \mathbf{y}_1$.

To this end, we notice that $\mathbf{y}_m^*$ for any $m = 2, \dots, T/K$ is equal to

$$\mathbf{y}_m^* = \operatorname{argmin}_{\mathbf{x} \in \mathcal{K}_\delta} \left\{ \left\langle \sum_{i=1}^{m-1} (\nabla_i - \alpha K \mathbf{y}_i), \mathbf{x} \right\rangle + \frac{\alpha(m-1)K}{2} \|\mathbf{x}\|_2^2 \right\}. \quad (30)$$

Similarly, for any $m = 2, \dots, T/K$, the action $\mathbf{y}_m$ of Algorithm 1 with $\alpha > 0$ is equal to

$$\mathbf{y}_m = \operatorname{argmin}_{\mathbf{x} \in \mathcal{K}_\delta} \left\{ \left\langle \bar{\mathbf{g}}_{(m-1)K} - \sum_{i=1}^{m-1} \alpha K \mathbf{y}_i, \mathbf{x} \right\rangle + \frac{\alpha(m-1)K}{2} \|\mathbf{x}\|_2^2 \right\}. \quad (31)$$

Combining (30) and (31) with Lemma 4, for any $m = 2, \dots, T/K$, we have

$$\|\mathbf{y}_m - \mathbf{y}_m^*\|_2 \leq \frac{1}{\alpha(m-1)K} \left\| \bar{\mathbf{g}}_{(m-1)K} - \sum_{i=1}^{m-1} \nabla_i \right\|_2 = \frac{1}{\alpha(m-1)K} \left\| \sum_{t \in \mathcal{U}_m} \frac{n}{\delta} f_t(\mathbf{x}_t) \mathbf{u}_t \right\|_2 \quad (32)$$

where $\mathcal{U}_m$ is defined in (18).

Moreover, from (1), for any $m = 2, \dots, T/K$, we have

$$\begin{aligned} \|\mathbf{y}_m^* - \mathbf{y}_{m+1}^*\|_2^2 &\leq \frac{2}{\alpha m K} \left(\sum_{i=1}^m \ell_i(\mathbf{y}_m^*) - \sum_{i=1}^m \ell_i(\mathbf{y}_{m+1}^*) \right) \\ &\leq \frac{2}{\alpha m K} (\ell_m(\mathbf{y}_m^*) - \ell_m(\mathbf{y}_{m+1}^*)) \\ &\leq \frac{2}{\alpha m K} (\langle \nabla_m + \alpha K(\mathbf{y}_m^* - \mathbf{y}_m), \mathbf{y}_m^* - \mathbf{y}_{m+1}^* \rangle) \\ &\leq \frac{2}{\alpha(m-1)K} (\|\nabla_m\|_2 + 2\alpha K R) \|\mathbf{y}_m^* - \mathbf{y}_{m+1}^*\|_2 \end{aligned}$$

where the first inequality is due to the definition of $\mathbf{y}_m^*$, and the last inequality is due to Assumption 1. The above inequality further implies that

$$\|\mathbf{y}_m^* - \mathbf{y}_{m+1}^*\|_2 \leq \frac{2}{\alpha(m-1)K} (\|\nabla_m\|_2 + 2\alpha K R). \quad (33)$$

Combining (27), (29), (32), and (33), we have

$$\begin{aligned} \mathbb{E}[\operatorname{Reg}(T)] &\leq \mathbb{E} \left[\sum_{m=1}^{T/K} \|\nabla_m\|_2 \|\mathbf{y}_m^* - \mathbf{y}_{m+1}^*\|_2 \right] + \sum_{m=1}^{T/K} K G \mathbb{E}[\|\mathbf{y}_m - \mathbf{y}_m^*\|_2] + 3\delta G T + \frac{\delta G R T}{r} \\ &\leq \mathbb{E}[\|\nabla_1\|_2 \|\mathbf{y}_1 - \mathbf{y}_2^*\|_2] + \mathbb{E} \left[\sum_{m=2}^{T/K} \frac{2(\|\nabla_m\|_2^2 + 2\alpha K R \|\nabla_m\|_2)}{\alpha(m-1)K} \right] \\ &\quad + \frac{G}{\alpha(m-1)} \sum_{m=2}^{T/K} \mathbb{E} \left[\left\| \sum_{t \in \mathcal{U}_m} \frac{n}{\delta} f_t(\mathbf{x}_t) \mathbf{u}_t \right\|_2 \right] + 3\delta G T + \frac{\delta G R T}{r}. \end{aligned}$$

Combining the above inequality with Assumption 1, Lemma 5, Lemma 6, and $\gamma = K \left(\frac{nM}{\delta} \right)^2 + K^2 G^2$, it is easy to verify that

$$\begin{aligned} \mathbb{E}[\operatorname{Reg}(T)] &\leq 2R\sqrt{\gamma} + \sum_{m=2}^{T/K} \frac{1}{m-1} \left(\frac{2\gamma}{\alpha K} + 4R\sqrt{\gamma} + \frac{G}{\alpha} \sqrt{2 \left(\frac{d^2}{K^2} + 4 \right) \gamma} \right) + 3\delta G T + \frac{\delta G R T}{r} \\ &\leq 2R\sqrt{\gamma} + (1 + \ln T) \left(\frac{2\gamma}{\alpha K} + 4R\sqrt{\gamma} + \frac{G}{\alpha} \sqrt{2 \left(\frac{d^2}{K^2} + 4 \right) \gamma} \right) + 3\delta G T + \frac{\delta G R T}{r} \\ &= \frac{2\gamma C_T}{\alpha K} + C'_T R \sqrt{\gamma} + \frac{G C_T}{\alpha} \sqrt{2 \left(\frac{d^2}{K^2} + 4 \right) \gamma} + 3\delta G T + \frac{\delta G R T}{r} \end{aligned}$$

where $C_T = 1 + \ln T$ and $C'_T = 6 + 4 \ln T$.

E Proof of Theorem 3

The main idea of this proof is to combine the proof of Theorem 2 with an improved property of the δ -smoothed version of smooth functions [Agarwal et al., 2010].

Specifically, for any $t \in [T]$ and $\mathbf{x}$, according to the smoothness of functions, we have

$$\hat{f}_{t,\delta}(\mathbf{x}) \leq \mathbb{E}_{\mathbf{u} \sim \mathcal{B}^n} \left[f_t(\mathbf{x}) + \langle \nabla f_t(\mathbf{x}), \delta \mathbf{u} \rangle + \frac{\beta \delta^2 \|\mathbf{u}\|_2^2}{2} \right] = f_t(\mathbf{x}) + \frac{\beta \delta^2}{2} \quad (34)$$

where $\hat{f}_{t,\delta}(\mathbf{x}) = \mathbb{E}_{\mathbf{u} \sim \mathcal{B}^n} [f_t(\mathbf{x} + \delta \mathbf{u})]$ and the last equality is due to $\mathbb{E}_{\mathbf{u} \sim \mathcal{B}^n} [\mathbf{u}] = \mathbf{0}$.

Moreover, due to the convexity of functions, for any $t \in [T]$ and $\mathbf{x}$, we have

$$\hat{f}_{t,\delta}(\mathbf{x}) \geq \mathbb{E}_{\mathbf{u} \sim \mathcal{B}^n} [f_t(\mathbf{x}) + \langle \nabla f_t(\mathbf{x}), \delta \mathbf{u} \rangle] = f_t(\mathbf{x}). \quad (35)$$

Then, let $\mathbf{x}^* = \operatorname{argmin}_{\mathbf{x} \in \mathbb{R}^n} \sum_{t=1}^T f_t(\mathbf{x})$ and $\tilde{\mathbf{x}}^* = (1 - \delta/r)\mathbf{x}^*$, where $r = 2G/\alpha$. According to (10), we have $\mathbf{x}^* \in \mathcal{K}'$ and $\tilde{\mathbf{x}}^* \in \mathcal{K}'_\delta$, where $\mathcal{K}'$ and $\mathcal{K}'_\delta$ are defined in (11) and (12), respectively. By further defining $\mathcal{T}_m = \{(m-1)K+1, \dots, mK\}$, it is not hard to verify that

$$\begin{aligned} \mathbb{E} [\operatorname{Reg}(T)] &= \mathbb{E} \left[\sum_{m=1}^{T/K} \sum_{t \in \mathcal{T}_m} (f_t(\mathbf{y}_m + \delta \mathbf{u}_t) - f_t(\mathbf{x}^*)) \right] \\ &\leq \mathbb{E} \left[\sum_{m=1}^{T/K} \sum_{t \in \mathcal{T}_m} \left(f_t(\mathbf{y}_m) + \langle \nabla f_t(\mathbf{y}_m), \delta \mathbf{u}_t \rangle + \frac{\beta \delta^2 \|\mathbf{u}_t\|_2^2}{2} \right) \right] \\ &\quad + \mathbb{E} \left[\sum_{m=1}^{T/K} \sum_{t \in \mathcal{T}_m} \left(-f_t(\tilde{\mathbf{x}}^*) + \left\langle \nabla f_t(\mathbf{x}^*), -\frac{\delta \mathbf{x}^*}{r} \right\rangle + \frac{\beta \delta^2 \|\mathbf{x}^*\|_2^2}{2r} \right) \right] \quad (36) \\ &= \mathbb{E} \left[\sum_{m=1}^{T/K} \sum_{t \in \mathcal{T}_m} \left(f_t(\mathbf{y}_m) - f_t(\tilde{\mathbf{x}}^*) + \frac{\beta \delta^2}{2} + \frac{\beta \delta^2 G}{\alpha} \right) \right] \\ &\leq \mathbb{E} \left[\sum_{m=1}^{T/K} \sum_{t \in \mathcal{T}_m} \left(\hat{f}_{t,\delta}(\mathbf{y}_m) - \hat{f}_{t,\delta}(\tilde{\mathbf{x}}^*) \right) \right] + \beta \delta^2 T + \frac{\beta \delta^2 GT}{\alpha} \end{aligned}$$

where the first inequality is due to the smoothness of functions, and the last inequality is due to (34) and (35).

Then, we follow the definition of $\mathbf{y}_m^*$ in (28), but replace $\mathcal{K}_\delta$ utilized in (28) with $\mathcal{K}'_\delta$. Combining (36) with the strong convexity of functions, we have

$$\begin{aligned} &\mathbb{E} [\operatorname{Reg}(T)] - \left(\beta \delta^2 T + \frac{\beta \delta^2 GT}{\alpha} \right) \\ &\leq \mathbb{E} \left[\sum_{m=1}^{T/K} \sum_{t \in \mathcal{T}_m} \left(\langle \nabla \hat{f}_{t,\delta}(\mathbf{y}_m), \mathbf{y}_m - \tilde{\mathbf{x}}^* \rangle - \frac{\alpha}{2} \|\mathbf{y}_m - \tilde{\mathbf{x}}^*\|_2^2 \right) \right] \\ &= \mathbb{E} \left[\sum_{m=1}^{T/K} \sum_{t \in \mathcal{T}_m} \left(\langle \nabla \hat{f}_{t,\delta}(\mathbf{y}_m), \mathbf{y}_m^* - \tilde{\mathbf{x}}^* \rangle + \langle \nabla \hat{f}_{t,\delta}(\mathbf{y}_m), \mathbf{y}_m - \mathbf{y}_m^* \rangle - \frac{\alpha}{2} \|\mathbf{y}_m - \tilde{\mathbf{x}}^*\|_2^2 \right) \right] \quad (37) \\ &\leq \mathbb{E} \left[\sum_{m=1}^{T/K} \sum_{t \in \mathcal{T}_m} \left(\langle \nabla \hat{f}_{t,\delta}(\mathbf{y}_m), \mathbf{y}_m^* - \tilde{\mathbf{x}}^* \rangle - \frac{\alpha}{2} \|\mathbf{y}_m - \tilde{\mathbf{x}}^*\|_2^2 \right) \right] + \mathbb{E} \left[\sum_{m=1}^{T/K} KG \|\mathbf{y}_m - \mathbf{y}_m^*\|_2 \right] \end{aligned}$$

where we simply set $\mathbf{y}_1^* = \mathbf{y}_1$, and the last inequality is due to Assumption 2 and the last property in Lemma 7.

Let $R = 2G/\alpha$ denote the radius of $\mathcal{K}'$. It is not hard to verify that (29), (32), and (33) in the proof of Theorem 2 still hold here. Therefore, we have

$$\begin{aligned}
& \mathbb{E}[\text{Reg}(T)] - \left(\beta\delta^2 T + \frac{\beta\delta^2 GT}{\alpha} \right) \\
& \leq \mathbb{E} \left[\sum_{m=1}^{T/K} \|\nabla_m\|_2 \|\mathbf{y}_m^* - \mathbf{y}_{m+1}^*\|_2 \right] + \mathbb{E} \left[\sum_{m=1}^{T/K} KG \|\mathbf{y}_m - \mathbf{y}_m^*\|_2 \right] \\
& \leq \mathbb{E}[\|\nabla_1\|_2 \|\mathbf{y}_1 - \mathbf{y}_2^*\|_2] + \mathbb{E} \left[\sum_{m=2}^{T/K} \frac{2(\|\nabla_m\|_2^2 + 2\alpha KR \|\nabla_m\|_2)}{\alpha(m-1)K} \right] \\
& \quad + \frac{G}{\alpha(m-1)} \sum_{m=2}^{T/K} \mathbb{E} \left[\left\| \sum_{t \in \mathcal{U}_m} \frac{n}{\delta} f_t(\mathbf{x}_t) \mathbf{u}_t \right\|_2 \right]
\end{aligned} \tag{38}$$

where the first inequality is due to (37) and (29), and the last one is due to (32) and (33).

Finally, combining (38) with Lemma 5, Lemma 6, and $\gamma = K \left(\frac{nM}{\delta} \right)^2 + K^2 G^2$, we have

$$\begin{aligned}
\mathbb{E}[\text{Reg}(T)] & \leq 2R\sqrt{\gamma} + \sum_{m=2}^{T/K} \frac{1}{m-1} \left(\frac{2\gamma}{\alpha K} + 4R\sqrt{\gamma} + \frac{G}{\alpha} \sqrt{2 \left(\frac{d^2}{K^2} + 4 \right) \gamma} \right) + \beta\delta^2 T + \frac{\beta\delta^2 GT}{\alpha} \\
& \leq 2R\sqrt{\gamma} + (1 + \ln T) \left(\frac{2\gamma}{\alpha K} + 4R\sqrt{\gamma} + \frac{G}{\alpha} \sqrt{2 \left(\frac{d^2}{K^2} + 4 \right) \gamma} \right) + \beta\delta^2 T + \frac{\beta\delta^2 GT}{\alpha} \\
& = \frac{2\gamma C_T}{\alpha K} + \frac{2C'_T G \sqrt{\gamma}}{\alpha} + \frac{GC_T}{\alpha} \sqrt{2 \left(\frac{d^2}{K^2} + 4 \right) \gamma} + \beta\delta^2 T + \frac{\beta\delta^2 GT}{\alpha}
\end{aligned}$$

where the last equality is due to $R = 2G/\alpha$, $C_T = 1 + \ln T$, and $C'_T = 6 + 4 \ln T$.

F A refined regret bound for Bistriz et al. [2022]

From Theorem 4 of Bistriz et al. [2022], their algorithm can achieve the following regret bound

$$\mathbb{E}[\text{Reg}(T)] = O \left(\delta T + \frac{\eta n^2 T}{\delta^2} + \frac{1}{\eta} + \frac{n\eta \bar{d} T}{\delta} \right) \tag{39}$$

for BCO with delayed feedback, where $\delta > 0$ and $\eta > 0$ denote the exploration radius and the step size, respectively. Then, by further substituting

$$\delta = \max \left\{ T^{-1/4}, T^{-1/3} \bar{d}^{1/3} \right\} \text{ and } \eta = \min \left\{ n^{-1} T^{-3/4}, n^{-1/2} T^{-2/3} \bar{d}^{-1/3} \right\}$$

into (39), Bistriz et al. [2022] have established the $O(nT^{3/4} + \sqrt{n} \bar{d}^{1/3} T^{2/3})$ regret bound. However, we notice that

$$\min_{\delta > 0, \eta > 0} O \left(\delta T + \frac{\eta n^2 T}{\delta^2} + \frac{1}{\eta} + \frac{n\eta \bar{d} T}{\delta} \right) = \min_{\delta > 0} O \left(\delta T + \sqrt{\frac{n^2 T}{\delta^2} + \frac{n \bar{d} T}{\delta}} \right) \tag{40}$$

where the equality holds with

$$\eta = \left(\frac{n^2 T}{\delta^2} + \frac{n \bar{d} T}{\delta} \right)^{-1/2}. \tag{41}$$

From (40), if $n^2 T \delta^{-2} \geq n \bar{d} T \delta^{-1}$, we have

$$\min_{\delta > 0, \eta > 0} O \left(\delta T + \frac{\eta n^2 T}{\delta^2} + \frac{1}{\eta} + \frac{n\eta \bar{d} T}{\delta} \right) = \min_{\delta > 0} O \left(\delta T + \frac{n\sqrt{T}}{\delta} \right) = O \left(\sqrt{n} T^{3/4} \right) \tag{42}$$

where the last equality holds with $\delta = \sqrt{n} T^{-1/4}$.

Otherwise, combining (40) with $n^2T\delta^{-2} < n\bar{d}T\delta^{-1}$, we have

$$\min_{\delta>0, \eta>0} O\left(\delta T + \frac{\eta n^2 T}{\delta^2} + \frac{1}{\eta} + \frac{n\eta\bar{d}T}{\delta}\right) = \min_{\delta>0} O\left(\delta T + \sqrt{\frac{n\bar{d}T}{\delta}}\right) = O\left((n\bar{d})^{1/3}T^{2/3}\right) \quad (43)$$

where the last equality holds with $\delta = (n\bar{d})^{1/3}T^{-1/3}$.

Combining (39) with (41), (42), and (43), we can improve the regret bound of Bistritz et al. [2022] to

$$\mathbb{E}[\text{Reg}(T)] = O\left(\sqrt{n}T^{3/4} + (n\bar{d})^{1/3}T^{2/3}\right) \quad (44)$$

by setting δ and η as

$$\delta = \max\left\{\sqrt{n}T^{-1/4}, (n\bar{d})^{1/3}T^{-1/3}\right\} \text{ and } \eta = \min\left\{n^{-1/2}T^{-3/4}, (n\bar{d})^{-1/3}T^{-2/3}\right\}.$$

NeurIPS Paper Checklist

1. Claims

Question: Do the main claims made in the abstract and introduction accurately reflect the paper's contributions and scope?

Answer: [\[Yes\]](#)

Justification: The abstract and introduction have clearly stated the contributions of this paper.

Guidelines:

- The answer NA means that the abstract and introduction do not include the claims made in the paper.
- The abstract and/or introduction should clearly state the claims made, including the contributions made in the paper and important assumptions and limitations. A No or NA answer to this question will not be perceived well by the reviewers.
- The claims made should match theoretical and experimental results, and reflect how much the results can be expected to generalize to other settings.
- It is fine to include aspirational goals as motivation as long as it is clear that these goals are not attained by the paper.

2. Limitations

Question: Does the paper discuss the limitations of the work performed by the authors?

Answer: [\[Yes\]](#)

Justification: In Appendix A, the authors discuss one limitation of the results, and introduce a related open problem.

Guidelines:

- The answer NA means that the paper has no limitation while the answer No means that the paper has limitations, but those are not discussed in the paper.
- The authors are encouraged to create a separate "Limitations" section in their paper.
- The paper should point out any strong assumptions and how robust the results are to violations of these assumptions (e.g., independence assumptions, noiseless settings, model well-specification, asymptotic approximations only holding locally). The authors should reflect on how these assumptions might be violated in practice and what the implications would be.
- The authors should reflect on the scope of the claims made, e.g., if the approach was only tested on a few datasets or with a few runs. In general, empirical results often depend on implicit assumptions, which should be articulated.
- The authors should reflect on the factors that influence the performance of the approach. For example, a facial recognition algorithm may perform poorly when image resolution is low or images are taken in low lighting. Or a speech-to-text system might not be used reliably to provide closed captions for online lectures because it fails to handle technical jargon.
- The authors should discuss the computational efficiency of the proposed algorithms and how they scale with dataset size.
- If applicable, the authors should discuss possible limitations of their approach to address problems of privacy and fairness.
- While the authors might fear that complete honesty about limitations might be used by reviewers as grounds for rejection, a worse outcome might be that reviewers discover limitations that aren't acknowledged in the paper. The authors should use their best judgment and recognize that individual actions in favor of transparency play an important role in developing norms that preserve the integrity of the community. Reviewers will be specifically instructed to not penalize honesty concerning limitations.

3. Theory Assumptions and Proofs

Question: For each theoretical result, does the paper provide the full set of assumptions and a complete (and correct) proof?

Answer: [Yes]

Justification: The assumptions of any theoretical result are clearly introduced in Section 3.3. The proofs for all theoretical results can be found in Section 3.4 and the appendix.

Guidelines:

- The answer NA means that the paper does not include theoretical results.
- All the theorems, formulas, and proofs in the paper should be numbered and cross-referenced.
- All assumptions should be clearly stated or referenced in the statement of any theorems.
- The proofs can either appear in the main paper or the supplemental material, but if they appear in the supplemental material, the authors are encouraged to provide a short proof sketch to provide intuition.
- Inversely, any informal proof provided in the core of the paper should be complemented by formal proofs provided in appendix or supplemental material.
- Theorems and Lemmas that the proof relies upon should be properly referenced.

4. Experimental Result Reproducibility

Question: Does the paper fully disclose all the information needed to reproduce the main experimental results of the paper to the extent that it affects the main claims and/or conclusions of the paper (regardless of whether the code and data are provided or not)?

Answer: [Yes]

Justification: The setup of our experiments is clearly introduced in Section 4, which is sufficient for reproducing our experimental results.

Guidelines:

- The answer NA means that the paper does not include experiments.
- If the paper includes experiments, a No answer to this question will not be perceived well by the reviewers: Making the paper reproducible is important, regardless of whether the code and data are provided or not.
- If the contribution is a dataset and/or model, the authors should describe the steps taken to make their results reproducible or verifiable.
- Depending on the contribution, reproducibility can be accomplished in various ways. For example, if the contribution is a novel architecture, describing the architecture fully might suffice, or if the contribution is a specific model and empirical evaluation, it may be necessary to either make it possible for others to replicate the model with the same dataset, or provide access to the model. In general, releasing code and data is often one good way to accomplish this, but reproducibility can also be provided via detailed instructions for how to replicate the results, access to a hosted model (e.g., in the case of a large language model), releasing of a model checkpoint, or other means that are appropriate to the research performed.
- While NeurIPS does not require releasing code, the conference does require all submissions to provide some reasonable avenue for reproducibility, which may depend on the nature of the contribution. For example
 - (a) If the contribution is primarily a new algorithm, the paper should make it clear how to reproduce that algorithm.
 - (b) If the contribution is primarily a new model architecture, the paper should describe the architecture clearly and fully.
 - (c) If the contribution is a new model (e.g., a large language model), then there should either be a way to access this model for reproducing the results or a way to reproduce the model (e.g., with an open-source dataset or instructions for how to construct the dataset).
 - (d) We recognize that reproducibility may be tricky in some cases, in which case authors are welcome to describe the particular way they provide for reproducibility. In the case of closed-source models, it may be that access to the model is limited in some way (e.g., to registered users), but it should be possible for other researchers to have some path to reproducing or verifying the results.

5. Open access to data and code

Question: Does the paper provide open access to the data and code, with sufficient instructions to faithfully reproduce the main experimental results, as described in supplemental material?

Answer: [No]

Justification: Any interested people can send the authors an email to query the source code.

Guidelines:

- The answer NA means that paper does not include experiments requiring code.
- Please see the NeurIPS code and data submission guidelines (<https://nips.cc/public/guides/CodeSubmissionPolicy>) for more details.
- While we encourage the release of code and data, we understand that this might not be possible, so “No” is an acceptable answer. Papers cannot be rejected simply for not including code, unless this is central to the contribution (e.g., for a new open-source benchmark).
- The instructions should contain the exact command and environment needed to run to reproduce the results. See the NeurIPS code and data submission guidelines (<https://nips.cc/public/guides/CodeSubmissionPolicy>) for more details.
- The authors should provide instructions on data access and preparation, including how to access the raw data, preprocessed data, intermediate data, and generated data, etc.
- The authors should provide scripts to reproduce all experimental results for the new proposed method and baselines. If only a subset of experiments are reproducible, they should state which ones are omitted from the script and why.
- At submission time, to preserve anonymity, the authors should release anonymized versions (if applicable).
- Providing as much information as possible in supplemental material (appended to the paper) is recommended, but including URLs to data and code is permitted.

6. Experimental Setting/Details

Question: Does the paper specify all the training and test details (e.g., data splits, hyperparameters, how they were chosen, type of optimizer, etc.) necessary to understand the results?

Answer: [Yes]

Justification: The setup of our experiments is clearly introduced in Section 4.

Guidelines:

- The answer NA means that the paper does not include experiments.
- The experimental setting should be presented in the core of the paper to a level of detail that is necessary to appreciate the results and make sense of them.
- The full details can be provided either with the code, in appendix, or as supplemental material.

7. Experiment Statistical Significance

Question: Does the paper report error bars suitably and correctly defined or other appropriate information about the statistical significance of the experiments?

Answer: [No]

Justification: We have run each algorithm 20 times, and reported the average result in Section 4. The standard deviation is omitted because we believe that it does not affect the comparison between our algorithm and baselines.

Guidelines:

- The answer NA means that the paper does not include experiments.
- The authors should answer "Yes" if the results are accompanied by error bars, confidence intervals, or statistical significance tests, at least for the experiments that support the main claims of the paper.
- The factors of variability that the error bars are capturing should be clearly stated (for example, train/test split, initialization, random drawing of some parameter, or overall run with given experimental conditions).

- The method for calculating the error bars should be explained (closed form formula, call to a library function, bootstrap, etc.)
- The assumptions made should be given (e.g., Normally distributed errors).
- It should be clear whether the error bar is the standard deviation or the standard error of the mean.
- It is OK to report 1-sigma error bars, but one should state it. The authors should preferably report a 2-sigma error bar than state that they have a 96% CI, if the hypothesis of Normality of errors is not verified.
- For asymmetric distributions, the authors should be careful not to show in tables or figures symmetric error bars that would yield results that are out of range (e.g. negative error rates).
- If error bars are reported in tables or plots, The authors should explain in the text how they were calculated and reference the corresponding figures or tables in the text.

8. Experiments Compute Resources

Question: For each experiment, does the paper provide sufficient information on the computer resources (type of compute workers, memory, time of execution) needed to reproduce the experiments?

Answer: [Yes]

Justification: As mentioned in Section 4, all algorithms are tested on a laptop with 2.4GHz CPU and 16GB memory.

Guidelines:

- The answer NA means that the paper does not include experiments.
- The paper should indicate the type of compute workers CPU or GPU, internal cluster, or cloud provider, including relevant memory and storage.
- The paper should provide the amount of compute required for each of the individual experimental runs as well as estimate the total compute.
- The paper should disclose whether the full research project required more compute than the experiments reported in the paper (e.g., preliminary or failed experiments that didn't make it into the paper).

9. Code Of Ethics

Question: Does the research conducted in the paper conform, in every respect, with the NeurIPS Code of Ethics <https://neurips.cc/public/EthicsGuidelines>?

Answer: [Yes]

Justification: The authors conducted the research in this paper by strictly conforming to the NeurIPS Code of Ethics.

Guidelines:

- The answer NA means that the authors have not reviewed the NeurIPS Code of Ethics.
- If the authors answer No, they should explain the special circumstances that require a deviation from the Code of Ethics.
- The authors should make sure to preserve anonymity (e.g., if there is a special consideration due to laws or regulations in their jurisdiction).

10. Broader Impacts

Question: Does the paper discuss both potential positive societal impacts and negative societal impacts of the work performed?

Answer: [NA]

Justification: This paper is mostly theoretical and the societal impacts discussion is not applicable.

Guidelines:

- The answer NA means that there is no societal impact of the work performed.
- If the authors answer NA or No, they should explain why their work has no societal impact or why the paper does not address societal impact.

- Examples of negative societal impacts include potential malicious or unintended uses (e.g., disinformation, generating fake profiles, surveillance), fairness considerations (e.g., deployment of technologies that could make decisions that unfairly impact specific groups), privacy considerations, and security considerations.
- The conference expects that many papers will be foundational research and not tied to particular applications, let alone deployments. However, if there is a direct path to any negative applications, the authors should point it out. For example, it is legitimate to point out that an improvement in the quality of generative models could be used to generate deepfakes for disinformation. On the other hand, it is not needed to point out that a generic algorithm for optimizing neural networks could enable people to train models that generate Deepfakes faster.
- The authors should consider possible harms that could arise when the technology is being used as intended and functioning correctly, harms that could arise when the technology is being used as intended but gives incorrect results, and harms following from (intentional or unintentional) misuse of the technology.
- If there are negative societal impacts, the authors could also discuss possible mitigation strategies (e.g., gated release of models, providing defenses in addition to attacks, mechanisms for monitoring misuse, mechanisms to monitor how a system learns from feedback over time, improving the efficiency and accessibility of ML).

11. Safeguards

Question: Does the paper describe safeguards that have been put in place for responsible release of data or models that have a high risk for misuse (e.g., pretrained language models, image generators, or scraped datasets)?

Answer: [NA]

Justification: This paper is mostly theoretical and poses no such risks.

Guidelines:

- The answer NA means that the paper poses no such risks.
- Released models that have a high risk for misuse or dual-use should be released with necessary safeguards to allow for controlled use of the model, for example by requiring that users adhere to usage guidelines or restrictions to access the model or implementing safety filters.
- Datasets that have been scraped from the Internet could pose safety risks. The authors should describe how they avoided releasing unsafe images.
- We recognize that providing effective safeguards is challenging, and many papers do not require this, but we encourage authors to take this into account and make a best faith effort.

12. Licenses for existing assets

Question: Are the creators or original owners of assets (e.g., code, data, models), used in the paper, properly credited and are the license and terms of use explicitly mentioned and properly respected?

Answer: [Yes]

Justification: In Section 4, we have indicated that data sets are selected from the LIBSVM repository [Chang and Lin, 2011].

Guidelines:

- The answer NA means that the paper does not use existing assets.
- The authors should cite the original paper that produced the code package or dataset.
- The authors should state which version of the asset is used and, if possible, include a URL.
- The name of the license (e.g., CC-BY 4.0) should be included for each asset.
- For scraped data from a particular source (e.g., website), the copyright and terms of service of that source should be provided.

- If assets are released, the license, copyright information, and terms of use in the package should be provided. For popular datasets, paperswithcode.com/datasets has curated licenses for some datasets. Their licensing guide can help determine the license of a dataset.
- For existing datasets that are re-packaged, both the original license and the license of the derived asset (if it has changed) should be provided.
- If this information is not available online, the authors are encouraged to reach out to the asset's creators.

13. **New Assets**

Question: Are new assets introduced in the paper well documented and is the documentation provided alongside the assets?

Answer: [NA]

Justification: This paper does not release new assets.

Guidelines:

- The answer NA means that the paper does not release new assets.
- Researchers should communicate the details of the dataset/code/model as part of their submissions via structured templates. This includes details about training, license, limitations, etc.
- The paper should discuss whether and how consent was obtained from people whose asset is used.
- At submission time, remember to anonymize your assets (if applicable). You can either create an anonymized URL or include an anonymized zip file.

14. **Crowdsourcing and Research with Human Subjects**

Question: For crowdsourcing experiments and research with human subjects, does the paper include the full text of instructions given to participants and screenshots, if applicable, as well as details about compensation (if any)?

Answer: [NA]

Justification: This paper does not involve crowdsourcing nor research with human subjects.

Guidelines:

- The answer NA means that the paper does not involve crowdsourcing nor research with human subjects.
- Including this information in the supplemental material is fine, but if the main contribution of the paper involves human subjects, then as much detail as possible should be included in the main paper.
- According to the NeurIPS Code of Ethics, workers involved in data collection, curation, or other labor should be paid at least the minimum wage in the country of the data collector.

15. **Institutional Review Board (IRB) Approvals or Equivalent for Research with Human Subjects**

Question: Does the paper describe potential risks incurred by study participants, whether such risks were disclosed to the subjects, and whether Institutional Review Board (IRB) approvals (or an equivalent approval/review based on the requirements of your country or institution) were obtained?

Answer: [NA]

Justification: This paper does not involve crowdsourcing nor research with human subjects.

Guidelines:

- The answer NA means that the paper does not involve crowdsourcing nor research with human subjects.
- Depending on the country in which research is conducted, IRB approval (or equivalent) may be required for any human subjects research. If you obtained IRB approval, you should clearly state this in the paper.

- We recognize that the procedures for this may vary significantly between institutions and locations, and we expect authors to adhere to the NeurIPS Code of Ethics and the guidelines for their institution.
- For initial submissions, do not include any information that would break anonymity (if applicable), such as the institution conducting the review.