# An Alternative Method for Determining Penetration Limit Velocities Using Residual Velocity Data

**K. Hohnecker**[1] and C. Drake[2]

[1]*DEVCOM Armaments Center, Picatinny Arsenal, NJ, USA*
*kenneth.h.hohnecker.civ@army.mil*
[2]*DEVCOM Armaments Center, Picatinny Arsenal, NJ, USA*
*christopher.g.drake4.civ@army.mil*

**Abstract.** Limit velocities are the impact velocities at which a penetrator has a certain probability of perforating a given target. These limit velocities are often used as performance metrics to evaluate the effectiveness of targets (e.g., personal armour) at stopping a given penetrator. Limit velocities typically need to be determined experimentally, especially for new designs or concepts for which there is little or no pre-existing data. When evaluating protection against small arms, these limit velocity tests often employ an adaptive binary data gathering algorithm. One issue encountered when modelling binary data (perforation/no-perforation in this case) is that one needs a relatively large sample size to develop a model with reasonable confidence bounds (precision) due to the information-sparse nature of binary data. In recent years, the ability to capture the residual velocity of these penetrators after impacting the target has become more prevalent through the use of high-speed cameras or other modern instrumentation. The new methodology outlined by the authors in this paper demonstrates that the inclusion of this additional continuous data significantly improves both precision and efficiency with regard to the modelling of limit velocities. This paper will discuss the development of the equation for residual velocities that was sufficiently generic to apply to a wide range of penetrators and targets, while also remaining amenable to a tractable and computationally efficient statistical analysis. The authors go on to demonstrate the improvement to efficiency and precision using various Monte Carlo and re-sampling comparisons to traditional binary testing and modelling methods.

## 1. BACKGROUND

Ballistic penetration testing is an integral part of research and development, as well as demonstration testing, for various military commodities including personnel protection, ammunition, safety equipment, and weaponry. The binary nature of perforation testing can pose some problems with regard to the inherent inefficiency and variation of the data. There are also risks when trying to find the area of overlap from perforation to no-perforation (Zone of Mixed Results [ZMR]): if this region is missed in the test data, modelling the data has traditionally not been feasible. In most scenarios where continuous responses can be measured (e.g., velocity), useful estimates of relevant parameters including mean and standard deviation can be derived from relatively small sample sizes. With binary data, proportions are the relevant parameters, and more samples are generally required to estimate a proportion with similar precision. When trying to model a continuous response, like velocity as a function of another factor (i.e., propellant charge), interpolative Least Squares Regression models can be easily generated with a small sampling of points along the predictor input space. For binary data, more samples are required to produce a similarly precise model, with Binary Logistic Regression with a Logit link function being the preferred modelling method [1].

Adaptive test algorithms (e.g., Langlie [2], 3-Phase Optimal Design [3POD] [3]) have helped to improve efficiency with respect to lowering sample sizes, but sample size demands still surpass that of continuous responses. In cases where no residual continuous response exists, 3POD has been shown to be the preferred binary data collection method [4] and has seen increased use in United States Department of Defense applications since its development. However, when relevant continuous response metrics can be measured concurrently with a binary result, the authors posit that improvements in efficiency and precision can be obtained compared to Binary Logistic Regression and 3POD using the modelling technique described in this paper, and a new testing algorithm currently in development by the authors.

## 2. METHODOLOGY

### 2.1 Derivation of Residual Velocity as a Function of Striking Velocity

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The derivation begins by assuming that, for given target and penetrator materials and geometry, the resistive pressure ($P$) acting on the penetrator depends on the penetrator's instantaneous velocity inside the target ($v$), a strength parameter ($S$, units of force per area), and a characteristic velocity ($v_c$). Using standard non-dimensionalisation techniques, one can write that

$$\frac{P}{S} = g\left(\frac{v}{v_c}\right), \tag{1}$$

where $g(\cdot)$ is some non-negative, dimensionless function. Since $P$ is a force per unit area, one can substitute Equation 1 into Newton's Second Law to obtain

$$\frac{dv}{dt} = -\frac{AS}{m} g\left(\frac{v}{v_c}\right), \tag{2}$$

where $m$ is the mass of the penetrator, $t$ is time, and $A$ is the effective area of the penetrator. Equation 2 holds as long as $A$, $S$, and $m$ remain constant throughout the impact event. This is not true in general, and one expects that the quantity $\frac{m}{AS}$ will depend on the initial conditions of the event (i.e., the striking velocity ($v_s$)) and how far into the event one is (i.e., $t$). The authors assume that the velocity of the penetrator is strictly decreasing in time so that there is a bijective mapping between $t$ and $v$. Therefore,[6]

$$\frac{m}{AS} = q(v, v_s) \tag{3}$$

for some non-negative function $q(\cdot)$.

Since the authors are interested in the demarcation between perforation and non-perforation events, one must be able to model the dynamics near $v = 0$. The authors assume that $g(\cdot)$ is a smooth function, that $g(0) = 0$ (i.e., no retardation force when $v = 0$), and that it is strictly increasing (i.e., resistive pressure strictly increases with increasing velocity). A permissible candidate function for $g(\cdot)$ is therefore[7]

$$g\left(\frac{v}{v_c}\right) = \beta\left(\frac{v}{v_c}\right)^{\alpha}, \ \alpha, \beta > 0. \tag{4}$$

Substituting Equations 3 and 4 into Equation 2 yields

$$\frac{dv}{dt} = -\frac{\beta\left(\frac{v}{v_c}\right)^{\alpha}}{q(v, v_s)}. \tag{5}$$

Separating variables, and integrating both sides from the beginning of the impact to when the penetrator comes to rest (assuming the target is thick enough) yields

$$t_f = \int_0^{v_s} \frac{q(v, v_s)\, dv}{\beta\left(\frac{v}{v_c}\right)^{\alpha}}, \tag{6}$$

where $t_f$ is the total time elapsed. Since the factors in this integrand do not change signs, one can use the mean value theorem to obtain

$$t_f = \frac{v_c^{\alpha}\tilde{q}(v_s)}{\beta}\int_0^{v_s} v^{-\alpha} dv = \frac{v_c^{\alpha}\tilde{q}(v_s)}{(1-\alpha)\beta} v_s^{1-\alpha}, \ 0 < \alpha < 1 \tag{7}$$

[6] The authors chose to represent $\frac{m}{AS}$, and not $\frac{AS}{m}$, as a function since, for high striking velocities, significant erosion of the penetrator may occur such that $m$ may approach 0.

[7] One reason for this choice is that $g(\cdot)$ cannot be represented by a power series near $v = 0$, as discussed later.

for some non-negative function $\tilde{q}(\cdot)$. Note that $\alpha$ must be constrained to less than unity to ensure that the penetrator stops in a finite amount of time.[8] With this additional constraint on $\alpha$, take Equation 5, divide both sides by $v$, note that $\frac{dv}{dt}/v = \frac{dv}{dx}$, separate variables, and integrate across the thickness of the target that is perforated ($T$) to obtain

$$T = \frac{v_c^{\alpha}}{\beta} \int_{v_r}^{v_s} q\left(v, v_s\right) v^{1-\alpha} dv, \; v_r \geq 0, \; 0 < \alpha < 1 \quad (8)$$

where $v_r$ is the residual velocity (i.e., exit velocity) of the penetrator.

The authors wished to be able to implement an estimation method to determine the parameters in Equation 8 that give the "best fit" to test data. These estimates could then be plugged back into Equation 8 to approximate limit velocities. Fitting the data is relatively straight forward if the penetrator perforates the target, since $v_s$, $v_r$, and $T$ are all known or can be readily measured. Also, knowledge of the dependency of $q(\cdot)$ on $v$ is no longer required, as the former can be pulled out of the integral (using the mean value theorem) and replaced with a function dependent only on $v_s$.

However, if perforation does not occur, then $T$ will not be known. One must then measure $T$ (which may be difficult, especially if the penetrator gets lodged in the target or if the target is littered with debris and fractures), and still one is left with determining the dependency of $q(\cdot)$ on $v$. Alternatively, one can throw out any test data where perforation does not occur (which is inefficient).

Therefore, this paper proposes the following method that ensures that all data is used in parameter estimation, without requiring the measuring of penetration depths. For non-perforating data, rather than setting $v_r = 0$ and measuring $T$, one leaves $T$ set to the target thickness and extends Equation 8 to allow for *negative* values of $v_r$. The authors now abandon the definition of $v_r$ as the physical residual velocity and instead think of it as a more abstract measure of the "unconsumed velocity" after penetrating a certain thickness of target. This definition still makes physical sense when the penetrator completely perforates a target. However, if complete perforation does not occur, and the penetrator only perforated a thickness $T_1$, then the amount of unconsumed velocity that is *lacking* (i.e., $-v_r$) to penetrate a second, adjacent target with thickness $T_2$ can be calculated from Equation 8 as[9]

$$T_2 = \frac{v_c^{\alpha}}{\beta} \int_{0}^{-v_r} q\left(-v, v_s\right) v^{1-\alpha} dv. \quad (9)$$

Therefore, if the thickness of the target that the penetrator actually perforates ($T_1$) is less than the total thickness ($T$), then one sets $T_2 = T - T_1$ and obtains an extension of Equation 8 to negative values of $v_r$, given by

$$T = \begin{cases} \frac{v_c^{\alpha}}{\beta} \int_{v_r}^{v_s} q\left(v, v_s\right) v^{1-\alpha} dv, & \frac{v_c^{\alpha}}{\beta} \int_{0}^{v_s} q\left(v, v_s\right) v^{1-\alpha} dv \geq T \\ \frac{v_c^{\alpha}}{\beta} \int_{0}^{v_s} q\left(v, v_s\right) v^{1-\alpha} dv + \frac{v_c^{\alpha}}{\beta} \int_{0}^{-v_r} q\left(-v, v_s\right) v^{1-\alpha} dv, & \text{otherwis} \end{cases}$$
$$= \frac{v_c^{\alpha}}{\beta} \int_{v_r}^{v_s} q\left(v, v_s\right) |v|^{1-\alpha} dv. \quad (10)$$

For a fixed $T$ and fixed target/penetrator properties, $v_r$ is strictly a function of $v_s$. Additionally, the factors of the integrand do not change signs; therefore, one can use the mean value theorem again to obtain

$$T = \frac{v_c^{\alpha}}{\beta} q_{eff}\left(v_s\right) \int_{v_r}^{v_s} |v|^{1-\alpha} dv, \quad (11)$$

where $q_{eff}(\cdot)$ is the effective value of $\frac{m}{AS}$ across the thickness of the target. The authors assume that $q_{eff}(\cdot)$ is an approximately linear function of $v_s$ near some striking velocity of interest ($v_\tau$), so that

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[8] Had $g\left(\frac{v}{v_c}\right)$ been expressed as a power series expansion about $\frac{v}{v_c} = 0$ with $g(0) = 0$, then, near $v = 0$, α would have effectively been greater than or equal to unity, implying that the penetrator would not come to rest in a finite amount of time. Since this is not physically realistic, the authors did not permit any function for $g(\cdot)$ that could be represented by a power series near $v = 0$.

[9] Here the authors have substituted $v_s$ in the upper integration limit of Equation 8 with $-v_r$, since this is now the "initial velocity" going into the second target. The lower integration limit of Equation 8 is set to zero as one is interested in how much unconsumed velocity is required to exactly penetrate the second target (i.e., when is the residual velocity out of the second target exactly zero). The authors have also included a minus sign in front of the $v$ inside of $q(\cdot)$ to indicate that this is the extension of $q$ to "negative velocities" given the true initial striking velocity of $v_s$. Note that $v_r < 0$ for $T_2 > 0$.


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$$q_{eff}(v_s) \approx Q_* \left[1 - c\left(\frac{v_s}{v_\tau} - 1\right)\right], \tag{12}$$

where $Q_*$ is the effective value of $\frac{m}{AS}$ across the target thickness at a striking velocity of $v_\tau$ and $c$ is some constant. Since $q_{eff}(v_s)$ is non-negative, one also wants the approximation to be non-negative at all values of $v_s$ under consideration. The authors assume that $Q_*$ is not zero, therefore imposing the constraint

$$1 - c\left(\frac{v_s}{v_\tau} - 1\right) \geq 0 . \tag{13}$$

Substituting Equation 12 into Equation 11 and solving for $v_r$ yields

$$\begin{cases} v_r &= \text{sign}(x)\,|x|^{\frac{1}{2-\alpha}}, \\ x &= v_s^{2-\alpha} - \frac{T\beta(2-\alpha)}{v_c^{\alpha} Q_* \left[1 - c\left(\frac{v_s}{v_\tau} - 1\right)\right]} \end{cases} \quad (14)$$

Observe that $v_c$ is just a constant with respect to a penetrator-target system and can be absorbed into $\beta$; therefore, $v_c$ is essentially arbitrary. For convenience, then, the authors set $v_c = v_\tau$, where $v_\tau$ is now defined to be the penetrator striking velocity at which the target is perforated with probability $\tau$ (that is, the limit velocity associated with probability $\tau$). Additionally, replace the expression $\frac{T\beta(2-\alpha)}{Q_*}$, which is non-negative and does not depend on $v_s$, with the variable $u^2$, so that Equation 14 simplifies to

$$\begin{cases} v_r &= \text{sign}(x)\,|x|^{\frac{1}{2-\alpha}}, \\ x &= v_s^{2-\alpha} - \frac{u^2}{v_\tau^{\alpha} \left[1 - c\left(\frac{v_s}{v_\tau} - 1\right)\right]} \end{cases} \quad (15)$$

Note that $u$ has units of velocity.

Lastly, given the intended use-case of modelling $v_r$ at a $v_s$ near the limit velocity, the authors expect that $v_r$ will be strictly increasing with respect to $v_s$ (i.e., $\frac{dv_r}{dv_s} > 0$), which yields the constraint

$$1 - \left(\frac{v_s}{v_\tau}\right)^{\alpha} \frac{cu^2}{(2-\alpha)\,v_s v_\tau \left[1 - c\left(\frac{v_s}{v_\tau} - 1\right)\right]^2} > 0 . \tag{16}$$

The authors thus arrived at their form for $v_r$ as a function of $v_s$, which is given by Equation 15 and subject to $0 < \alpha < 1$, Equation 13, and Equation 16. Note that caution must be taken if Equation 15 is used to model $v_r$'s beyond its intended purpose of finding limit velocities.

**2.2 Probability Distribution of the Residual Velocity**

The authors now wish to approximate the probability distribution of the residual velocity as a function of striking velocity. Upon examination of Equation 15, this paper proposes that the majority of shot-to-shot variation in residual velocity will result from the variation in $u$. A physical justification for this is that $u$ contains variables related to the angle of attack of the penetrator at impact, the mass erosion of the penetrator, and the strength parameter of the penetrator/target interaction (via $Q_*$, $T$, and $\beta$), which the authors expect to be the more dominant stochastic processes in the impact event when compared to the "shape" of the velocity decay curve (defined by $\alpha$) and the second order interactions with $v_s$ (defined by $c$). A heuristic mathematical justification is that, for $v_s$'s near the limit velocity of interest ($v_\tau$) and after factoring out $v_s^{2-\alpha}$, α will be the exponent of a number close to unity, and the term containing $c$ will be small compared to unity, so that the variations in either $\alpha$ or $c$ will make small changes to the value of $v_r$. Thus, an approximate distribution for the random variable (r.v.) of the residual velocity ($V_r$) is given by:

$$\begin{cases} V_r \mathrel{\dot\sim} \text{sign}(X)\,|X|^{\frac{1}{2-\alpha}}, \\ X \mathrel{\dot\sim} v_s^{2-\alpha} - \frac{U^2}{v_\tau^{\alpha} \left[1 - c\left(\frac{v_s}{v_\tau} - 1\right)\right]} \end{cases} \quad (17)$$

,

where the capital letters $X$ and $U$ are used to denote r.v. representations of $x$ and $u$, respectively.

One still needs to determine a probability distribution for $U$. Recall that $U$ represents a product of non-negative r.v.'s; therefore, $\ln(U)$ is a sum of r.v.'s. The Lyapunov variant of the Central Limit Theorem can be used to show that the sum of independent (but not necessarily identical) r.v.'s asymptotically approaches a normal distribution as the number of r.v.'s increases, assuming the higher moments of the individual r.v.'s are not "much bigger" than their variance ([5], [6]). Thus, the authors assume that $\ln(U)$ is approximately normally distributed, or equivalently, that $U$ is approximately log-normally distributed with parameters $\mu$ and $\sigma$, which are the mean and standard deviation, respectively, of $\ln(U)$.

The authors previously defined $v_\tau$ to be the striking velocity at which the penetrator will perforate the target with probability $\tau$. In order for this to hold, one must have that

$$\tau = P\left(v_r > 0 \mid v_s = v_\tau\right) = P\left(v_\tau^{2-\alpha} - \frac{U^2}{v_\tau^{\alpha}} > 0\right) = P\left(U < v_\tau\right) = F_U\left(v_\tau\right), \tag{18}$$

where Equation 17 has been used, and where $F_U(\cdot)$ is the cumulative distribution function (CDF) of $U$. Because of the log-normal approximation of $U$, one then has that

$$\begin{aligned} \tau &= \frac{1}{2}\left[1 + \operatorname{erf}\left(\frac{\ln v_\tau - \mu}{\sigma\sqrt{2}}\right)\right] \\ \implies \mu\left(v_\tau, \sigma\right) &= \ln v_\tau + \sigma\sqrt{2}\cdot \operatorname{erfc}^{-1}\left(2\tau\right), \end{aligned} \tag{19}$$

where $\operatorname{erfc}^{-1}(\cdot)$ is the inverse of the complementary error function. Therefore, the parameter $\mu$ is not free but is in fact a function of $v_\tau$ and $\sigma$.

### 2.3 Maximum Likelihood Estimate of Limit Velocity

The authors chose to use the maximum likelihood estimate (MLE) to determine the "best fit" parameters for Equation 17 given test data. The main idea behind the MLE is to create a "likelihood" function ($L$) that is the joint probability density function (PDF) of the $m$ observed values of $V_r$ (denoted $v_{r,1}, v_{r,2}, \ldots, v_{r,m}$) for a given a set of parameters. This likelihood function is then maximised with respect to the model parameters [6]. In other words, these maximising parameters ensure that the modelled probability distribution has the highest probability of drawing the observed data. If one assumes that the tests are independent, then $L$ is simply the product of the PDFs of $V_r$ for each data point.

In order to construct a likelihood function for the test data, therefore, one must first derive the CDF and PDF of $V_r$. Let $F_{Vr}$ be the CDF of $V_r$. With the help of Equation 17, one obtains, after some rearranging:

$$F_{Vr}(v_r; v_s, v_\tau, \alpha, c, \mathbb{U}) = P\left(V_r < v_r \mid v_s, v_\tau, \alpha, c, \mathbb{U}\right) = 1 - F_U\left[G(v_s, v_\tau, \alpha, c) M(v_r, v_s, \alpha)\right], \tag{20}$$

where

$$G\left(v_s, v_\tau, \alpha, c\right) = v_s\sqrt{\left(\frac{v_\tau}{v_s}\right)^{\alpha}\left[1 - c\left(\frac{v_s}{v_\tau} - 1\right)\right]}, \quad M\left(v_r, v_s, \alpha\right) = \sqrt{1 - \operatorname{sign}\left(v_r\right)\left(\frac{|v_r|}{v_s}\right)^{2-\alpha}}, \tag{21}$$

and $\mathbb{U}$ is, in general, the set of parameters that defines the distribution of $U$; for the authors' particular assumption that $U$ is log-normally distributed, $\mathbb{U} = \sigma$.[10] The PDF of $V_r$ is found by taking the derivative of $F_{Vr}(\cdot)$ with respect to $v_r$:

$$\therefore f_{V_r}\left(v_r; v_s, v_\tau, \alpha, c, \mathbb{U}\right) = \frac{(2-\alpha)\, G\left(v_s, v_\tau, \alpha, c\right)}{2 v_s M\left(v_r, v_s, \alpha\right)}\left(\frac{|v_r|}{v_s}\right)^{1-\alpha} f_U\left[G\left(v_s, v_\tau, \alpha, c\right) M\left(v_r, v_s, \alpha\right)\right]. \tag{22}$$

[10] Recall that $\mu$ is not a free parameter.


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With Equation 22 derived, one can now construct the likelihood function for the parameters $\alpha$, $c$, $v_\tau$, and $\mathbb{U}$.

Here one again runs into an issue with their test data when perforation does not occur. If $v_{r,i} > 0$, then the PDF of $V_r$ for the $i$th test instance is given by Equation 22. However, if the penetrator does not perforate, there is no means on knowing how "negative" $v_{r,i}$ is,[11] only that it is non-positive. To address this problem, the authors treat the $V_r$ r.v.'s for non-perforating data points as Bernoulli r.v.'s, where $v_{r,i}$ has a probability $p_i$ of being less than or equal to zero. One can calculate $p_i$ directly from Equation 20. Thus, the likelihood function given test data $\overrightarrow{v_s} = \left(v_{s,1}, v_{s,2}, \dots, v_{s,m}\right)^T$ and $\overrightarrow{v_r} = \left(v_{r,1}, v_{r,2}, \dots, v_{r,m}\right)^T$ is:[12]

$$L\left(v_\tau, \alpha, c, \mathbb{U}; \vec{v}_s, \vec{v}_r\right) = \prod_{\substack{i=1 \\ v_{r,i}=0}}^{m} F_{V_r}\left(0; v_{s,i}, v_\tau, \alpha, c, \mathbb{U}\right) \cdot \prod_{\substack{i=1 \\ v_{r,i}>0}}^{m} f_{V_r}\left(v_{r,i}; v_{s,i}, v_\tau, \alpha, c, \mathbb{U}\right) \quad . \qquad (23)$$

Thus, the authors seek to find the parameters that maximise $L$; that is, the MLE parameters.[13] The $v_\tau$ obtained as part of this maximising set of parameters (denoted $\hat{v}_r$) will therefore be the "best guess" of the limit velocity associated with probability $\tau$.

### 2.4 Construction of Confidence Bounds Around $\hat{v}_r$

To improve the usefulness of the approximation $\hat{v}_r$, one needs to construct upper and lower bounds on the estimate. Theoretically, the true $v_\tau$ will reside within this interval with some specified probability (often referred to in percentage as the confidence level [$CL$]).

Due to the complexity of the probability distribution of $V_r$ (nonstandard distribution and, under the authors' assumptions, four unknown parameters: $v_\tau$, $\alpha$, $c$, and $\sigma$) and the sparsity of data (it is not uncommon for a penetration test to consist of only 10 - 15 data points), the authors have chosen to use parametric bootstrapping to estimate the confidence bounds. The general procedure to find the upper confidence bound is as follows (a similar approach can be used to find the lower confidence bound) [6]:

1. Compute the MLE parameters ($\hat{v}_r, \hat{\alpha}, \hat{c}, \hat{\sigma}$) using the approach discussed in Section 2.3. This set of parameters is called the alternative hypothesis ($H_1$).
2. Make a guess for the upper bound of $v_\tau$ (denoted $v_{\tau,u}$).
3. Compute the MLE for the remaining three parameters assuming $v_{\tau,u}$ is true. This set of parameters (including $v_{\tau,u}$) is called the null hypothesis ($H_0$).
4. Randomly simulate many repetitions of the penetration test (at the same striking velocities as the data) using Equation 17 and the $H_0$ parameters;[14] use the results to approximate the distribution of some test statistic. The authors propose using the likelihood ratio test statistic ($\lambda_{LR}$) with $v_\tau = v_{\tau,u}$ as the null hypothesis <u>in computing $\lambda_{LR}$</u>. This is therefore an approximation of the distribution of $\lambda_{LR}$ assuming $H_0$ is true.
5. Using the simulated distribution, determine the (100 - $CL_u$)th percentile of $\lambda_{LR}$ (denoted $\tilde{\lambda}_{LR}$), where $CL_u$ is the confidence level of the upper bound.
6. Compute $\lambda_{LR}$ using the $H_1$ and $H_0$ parameters (denote this as $\lambda^*_{LR}$). If $\tilde{\lambda}_{LR} > \lambda^*_{LR}$, increase the guess for $v_{\tau,u}$; otherwise, decrease the guess for $v_{\tau,u}$.
7. Repeat steps 3 - 6 until $\tilde{\lambda}_{LR} = \lambda^*_{LR}$, to within some tolerance.

## 3. EXPERIMENTAL TESTING AND NUMERICAL SIMULATIONS

As an initial exploration of the validity of the method proposed in this paper, two pre-existing sets of perforation data (with accompanying residual velocity data) were obtained. One data set was of a small arms projectile against a "soft" metallic plate, and the second set of data was of the same projectile against a "hard" metallic plate. The $v_s$ vs. $v_r$ data is graphically depicted in Figure 1. These metallic data

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[11] Recall the generalised definition of $v_r$.

[12] Here, $T$ is the transpose operator, not target thickness.

[13] For computational purposes, a common practice is to maximise the log-likelihood function ($\ln(L)$) instead of $L$. This transformation has the benefit of turning multiplications into summations.

[14] To be compatible with the test data, simulations of striking velocities that did not perforate during testing should be modelled as binary r.v.'s (i.e., perforation/no-perforation), while simulations of striking velocities that did penetrate during testing should be modelled as continuous r.v.'s. Also note that for the latter simulations it is possible (and acceptable) to generate negative $v_r$'s.

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sets were investigated first due to their immediate availability; however, the authors plan on performing a similar analysis on other target materials, such as personal armour, once funding and testing logistics can be arranged.

The first step in the authors' evaluation was to fit the entire population of the data using the method discussed in Section 2.3 for both data sets. Due to the large number of data points, they therefore assumed that these fitted parameters were reasonable approximations to the population parameters. Next, using the fitted parameters, the authors graphed the Q-Q plots of $\ln(u)$ (calculated using Equation 15) against a normal distribution, which are shown in Figure 2. The closer the dots lie on a straight line, the more the $u$'s follow a log-normal distribution. Except for a few outliers near the tails, the distribution of the $u$'s appear to closely resembles a log-normal distribution for both data sets.

The second step in the evaluation process was to measure the precision of the proposed residual velocity method compared to logistic regression, which is a commonly used method in the estimation of limit velocities in the small arms field. For each of the data sets, evenly spaced striking velocities were chosen that roughly spanned the data. Penetration "tests" were simulated by generating residual velocities using the "population" parameters calculated previously, in conjunction with Equation 17.[15] A limit velocity was then estimated for each test. This process was repeated many times and the relative mean square error (RMSE) was computed for each of these tests when compared to the "true" limit

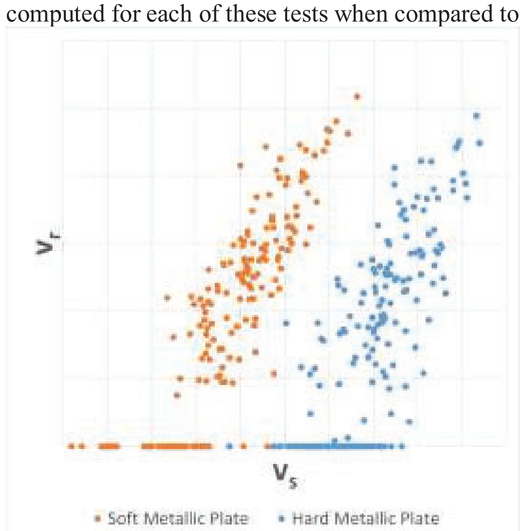


**Figure 1.** $v_s$ vs. $v_r$ curves of two penetration data sets

velocity. Figure 3 shows the plots of the RMSE when estimating the V10, V50, and V90 limit velocities as a function of the number of shots used in the simulated test. Observe that for V50 estimates for both the soft and hard plate, the proposed residual velocity method and logistic regression are comparably precise. However, when trying to estimate V10 or V90, the proposed method is significantly more precise than logistic regression, on the order of 1.5%. To look at it another way, the proposed method only requires 10 shots in a test to have the equivalent precision of a 25-shot test using logistic regression.

The final step was to evaluate the coverage of the confidence intervals[16] (CI) around the limit velocity estimates. That is, if one computes a 90% confidence interval, does the population limit velocity actually lie within the confidence interval with probability 0.9? Figure 4 shows the coverage of both the proposed method and logistic regression when computing confidence intervals for V10, V50, and V90 estimates when a 90% confidence interval is requested. Observe that logistic regression generally overshoots the desired coverage. At first, this result may seem to favor logistic regression; however, what this implies that logistic regression will, on average, construct confidence intervals that are larger than necessary. Figure 5 shows the average confidence interval widths for both the proposed method and logistic regression.

Note that logistic regression's confidence interval widths are on the order of one-and-a-half to two times as large as the proposed method. Consequently, when you evaluate data using logistic regression, you will have significantly less confidence in your results than if you evaluated the same data using the proposed method, potentially by a factor of 1.5-2.

[15] Assuming this distribution is justified based on the results of the Q-Q plots.
[16] Confidence interval is the interval bounded by the lower confidence bound and the upper confidence bound.

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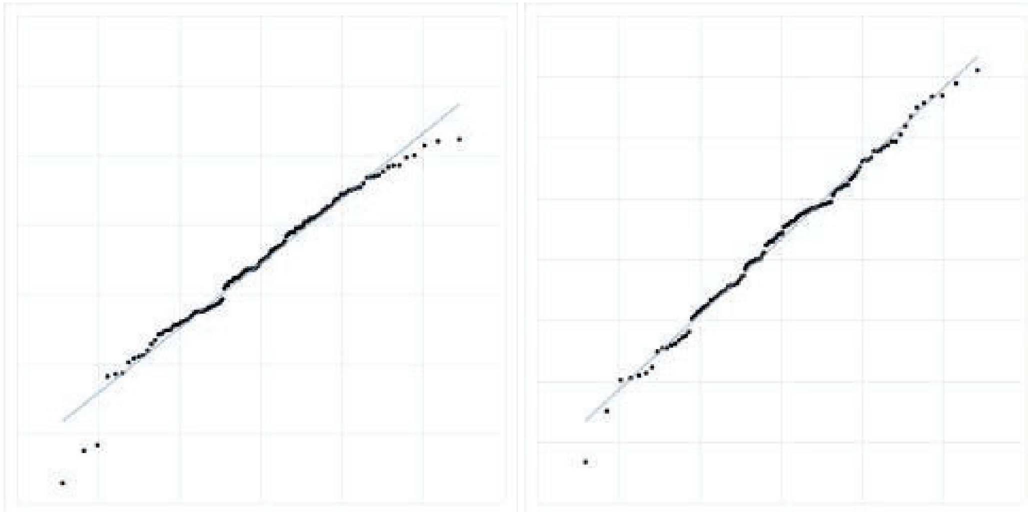

**Figure 2.** Q-Q Plots of ln($u$) against a normal distribution for soft (left) and hard (right) metallic plate data

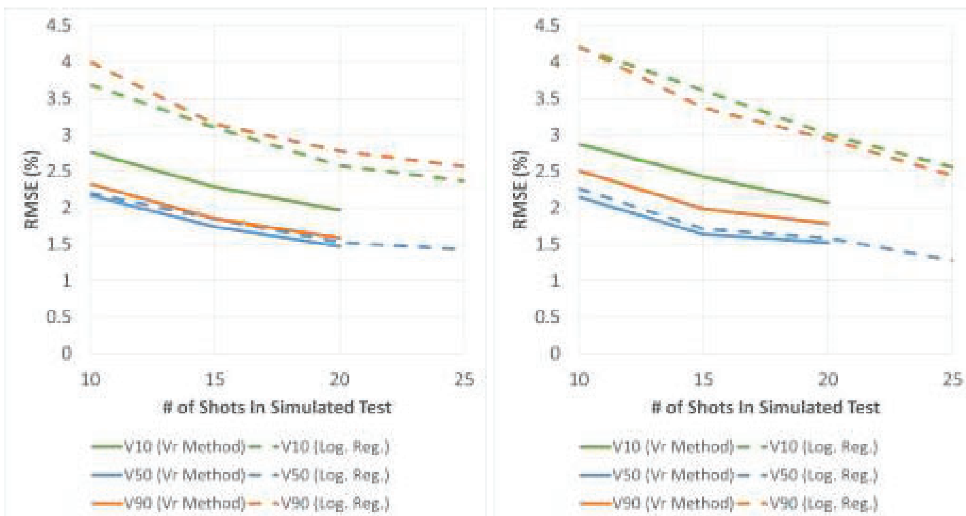


**Figure 3.** RMSE of limit velocity estimates vs. # of rounds in simulated tests for soft (left) and hard (right) metallic plate data

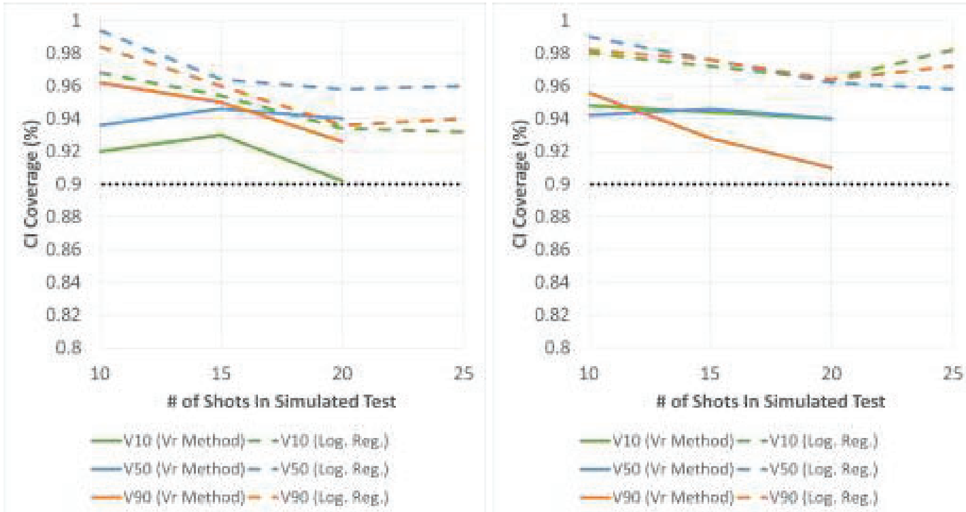


**Figure 4.** 90% CI coverage vs. # of rounds in simulated tests for soft (left) and hard (right) metallic plate data

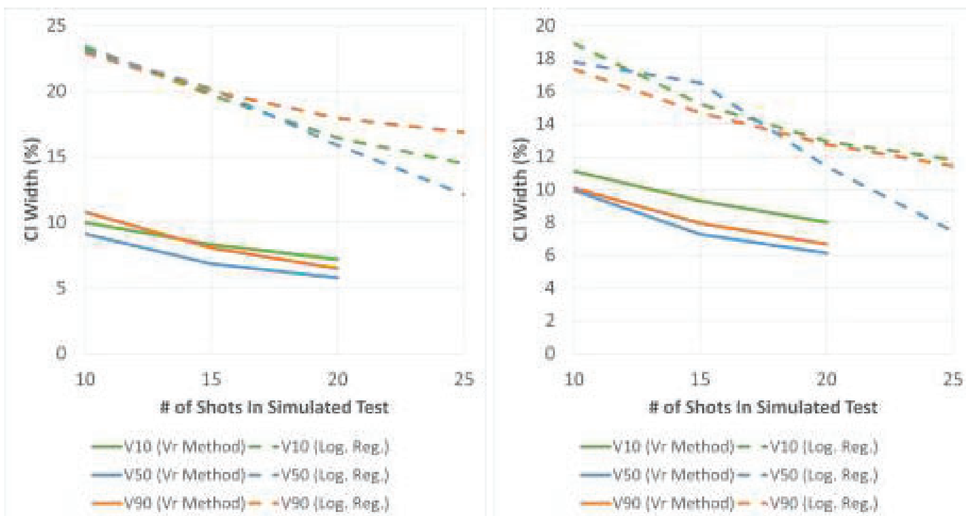


**Figure 5.** Avg. width of estimated 90% CI vs. # of rounds in simulated tests for soft (left) and hard (right) metallic plate data

## 4. FUTURE WORK

### 4.1 Live Fire Testing and Validation

The authors plan to further validate the limit velocity estimation method proposed by this paper. This will include augmenting the current data set by testing simple geometric penetrators against a wider range of targets, to include metals and ceramics, as well as any other materials deemed appropriate. Additionally, the authors plan on testing legacy military ammunition against various personal armour targets, to determine if the proposed methodology is robust enough to handle other complex dynamical interactions.

### 4.2 Test Algorithm Development

The development of the sister test algorithm to the modelling technique described in this paper is still in its early stages, but a few key differences from currently used adaptive test methods are noteworthy. In


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3POD and other traditional binary test methods, testing requires the observation of both perforation and no-perforation results to “home in” on the velocity region of interest. If during testing only perforations or no-perforations are observed, nothing of real use can be done with this data. Due to the nature of the modelling technique described in this paper, any no-perforation is essentially a sub-optimal data point, where no continuous residual data can be gleaned. This means that the test algorithm in development will focus on adaptively approaching the point at which residual velocity is estimated to reach zero without going over, starting at higher striking velocities and moving to lower striking velocities. Optimally efficient placement of test points to formulate the model will likely be along high leverage inflection points on the logistic regression curve. The details with regard to the desired spacing, starting point, and number of samples are still in development.

## 5. CONCLUSION

In this paper, the authors derived and formalised a new methodology for estimating limit velocities by analysing residual velocity data. They also demonstrated via test data and numerical simulation that the inclusion of this additional continuous data significantly improves both precision and efficiency with regard to the estimation of limit velocities with respect to the metallic targets analysed.

The derivations in this paper were based on material-agnostic principles; mainly, Newton’s Law, smoothness and monotonicity of the velocity retardation function, zero retardation force at termination of transient, and existence of a Taylor series expansion about the limit velocity. Thus, while the test data analysed in this paper demonstrated applicability to simple metallic targets, the authors posit that the methodology is sufficiently generic to apply to a wide range of penetrators and targets, to include personal armour targets.

Future work was discussed that will seek to continue to validate and refine the methodology, as well as investigate adaptive testing methods to further increase testing efficiency.

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